MODULE – 5: Introduction to Quantum Mechanics & Quantum Computing

Quantum Mechanics: Heisenberg Uncertainty principle and its significance, wave function- its significance, Dirac notation and Normalization of wave function, Schrodinger wave equation (qualitative), Application of Schrödinger wave equation for a particle in 1D infinite height potential well: Eigen states and Eigen values.– Numerical Problems

Principles of Quantum Information & Quantum Computing: Introduction, Moore's Law, Differences between Classical & Quantum computing. Qubit and its properties. Representation of qubit as Bloch sphere. Types of Qubits. –Single and Two qubits with one transformation operation for each.

Quantum mechanics is the branch of physics that deals with the **description of the physical properties of nature at the scale of atoms and subatomic particles** incorporating the concepts of quantization of energy, wave-particle duality, and the uncertainty principle. It differs from classical physics in that energy, momentum, and other quantities of a bound system are restricted to discrete values (quantization), objects have characteristics of both particle and waves (wave-particle duality); and there are limits to how accurately the value of a physical quantity can be predicted prior to its measurement, given a complete set of initial conditions (the uncertainty principle). It is the foundation of all quantum physics including quantum chemistry, quantum field theory, quantum technology, quantum information science, and quantum computing.

Heisenberg Uncertainty principle

According to de-Broglie, a moving material particle is associated with a wave packet, which has a small spread in the space. This wave nature implies that there are fundamental limits to the accuracy with which we can measure particle properties such as position and momentum.

Consider the wave group shown below fig (a)

The particle corresponding to this wave group may be located anywhere within the group at a given time. The narrower its wave group, the more precisely a particles position can be specified (fig (b)). But the wavelength λ of the waves in a narrow wave packet is not well defined. This means that, since $\lambda = h/p$, the particle's momentum is not a precise quantity.

On the other hand, a wide wave packet (fig (a)) has a clearly defined wavelength. The momentum that corresponds to this wavelength is therefore a precise quantity. But the width of the group is too large for us to be able to say exactly where the particle is at a given time. Thus, we have the uncertainty principle given by Heisenberg.

Position-Momentum Uncertainty principle

“**It is impossible to know both the exact position and exact momentum of an object at the same time.**”

Mathematically $\Delta x . \Delta p \geq \frac{h}{4\pi}$

Where Δx → uncertainty in the position, Δp → uncertainty in the momentum.

If we arrange matter, so that Δx is small, corresponding to narrow wave group then Δp will be large. If we reduce Δp in some way, a broad wave group is inevitable and Δx will be large.

Energy-Time uncertainty principle:

It states that “**It is impossible to measure both the exact energy and exact time in a physical process at the same time”**

i.e., $\Delta E . \Delta t \geq \frac{h}{4\pi}$

Where ΔE → uncertainty in the energy, Δt → uncertainty in the time

Angular displacement and Angular momentum uncertainty principle:

It states that “It states that “**It is impossible to measure both the exact angular displacement and exact angular momentum in a physical process at the same time**”

i.e., $\Delta L . \Delta \theta \geq \frac{h}{4\pi}$

Where ΔL → uncertainty in the momentum, $\Delta \theta$→ uncertainty in the displacement.

Physical Significance:

The physical significance of the Heisenberg’s uncertainty principle is that one should not think of the exact position or an accurate value for momentum of a particle. Instead, one should think of the probability of finding the particle at a certain position or of the most probable value for the momentum of the particle. Similar interpretation is made for the conjugate pair ΔE and Δt and $\Delta L$and $\Delta \theta$.

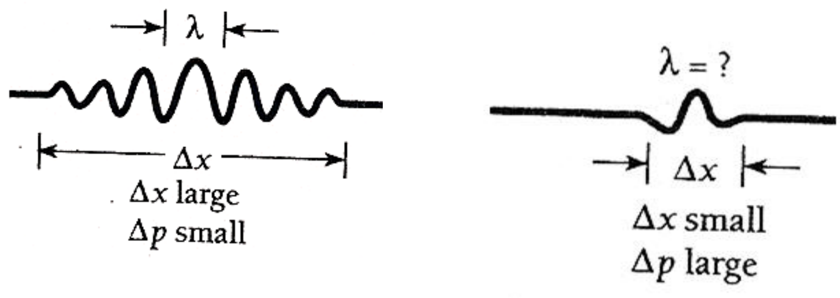


Fig. a

Wave function- its significance

Quantity whose variations make up matter waves is called the wave function. Exhibits wave-like properties of a particle and contain all possible information about the state of the system. The value of ψ associated with a moving body at x, y, z in space at the time 't' is related to the likelihood of finding the body there at that time. It is complex number with both real and imaginary parts.

Physical significance of Wave function

Probability of occurrence is a real and a positive quantity. Since ψ is complex, it has no attributable physical significance.

But the square of the absolute value of the wave function is always positive and real and is known as **probability density.**

It is given by $\mathbf{I\,\psi\,I^{2} = \psi.\ \Psi^{*}}$

where **Ψ*** is the complex conjugate of ψ.

The probability of finding particle at certain place at given time must be in between **0 and 1.** ex: 0.3 means probability of finding particle is 30% similarly -0.2 means meaningless. "The probability of experimentally finding a particle is proportional to the value of $\mathbf{I\psi I^2}$ there at 't'. Large value of $\mathbf{I\psi I^2}$ means the strong possibility of the body's presence, while a small value of means the slight-possibility of its presence.

Dirac notation

Normalization of Wave function

If ψ is the wave function associated with a particle, then the probability of **finding the particle in a volume dV is IψI^2 dV**. If the particle is present somewhere in a particular region, the integral of IψI^2 over all the space must be finite. If the particle is certainly to be found, in certain region of space then

$$\int_{-\infty}^{+\infty} |\psi^2| dV = 1$$

Any wave function satisfying the above equation is said to be **normalized wave function**. Often ψ is not a normalized wave function and have some constants. The process to determine that constant value and equate to ψ for unity is known as normalization

Eigen functions

Wave functions which are single valued with finite value everywhere and also their first derivatives with respect to their variables are continuous everywhere are known as **Eigen functions**

Eigen Values

The values of a physical observable such as energy, momentum etc for which Schrödinger's wave equation can be solved are called **Eigen values**.

<u>Schrodinger wave equation:</u>

Consider a particle, moving freely in the positive x-direction in a stationary potential field. The wavelength of the associated de-Broglie wave is given by

$$\lambda = \frac{h}{p} \qquad \rightarrow *$$

The wave equation for de-broglie wave associated with such particle can be written in complex notation as

$$\psi = Ae^{-i(wt-kx)}$$

A → Amplitude of the wave

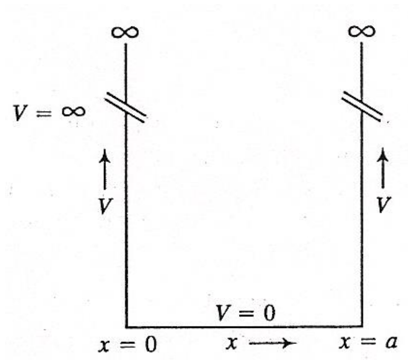
The time independent Schrödinger's wave equation for such particle is given by

$$\boxed{\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0}$$

Where E and V are total energy and potential energy of the particle respectively

Applications of Schrödinger's wave equation:

Energy Eigen values of a particle in one dimensional, infinite potential well (particle in a box):



Consider a particle, which is free to move in the x-direction only in the region x = 0 and x = L. Outside this region the potential energy is taken to be infinite and within this region it is zero ie V = 0 for 0<x<L and $V=\infty$ for $x \geq \infty$ and $x \leq 0$

We have the Schrödinger's time independent wave equation in one dimension

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}(E-V)\psi=0 \quad \text{------------ (1)}$$

Outside the well, the equation (1) become

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}(E-\infty)\psi=0 \qquad \text{Since } V=\infty$$

This equation holds good only if ψ = 0 for all points outside the well, ie $|\psi|^2=0$ which means that the particle cannot be found at all outside the well.

Inside the well, equation (1) becomes

$$\frac{d^2\psi}{dx^2}+\frac{8\pi^2 m}{h^2}E\psi=0 \quad \text{(Since V =0)}$$

Let $\frac{8\pi^2 m}{h^2}E=k^2$ ---------------- (2)

Then;

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \text{------------- (3)}$$

The solution for the above equation is

$$\psi = A\sin kx + B\cos kx \quad \text{------------- (4)}$$

At x = 0, ψ = 0, substituting in (4) we get

$$0 = A\sin 0 + B\cos 0$$

$$\Rightarrow B = 0$$

At x=L, ψ = 0 and equation (4) becomes

$$0 = A\sin kL + B\cos kL$$

$$\Rightarrow A\sin kL = 0 \qquad \text{(Since B=0)}$$

Here 'A' need not be zero

$$\therefore \qquad \sin kL = 0$$

i.e. $kL = n\pi$, where n=0, 1, 2, 3………..is an integer called quantum number.

$$\therefore \qquad k = \frac{n\pi}{L}$$

Substituting the values of B and k in (4)

$$\boxed{\psi_n = A\sin\frac{n\pi}{L}x} \quad \text{------------------ (5)}$$

which represents the permitted solutions.

Since there is only one particle and at any time it is present somewhere inside the well only, the integral of the wave function over the entire space in the well must be equal to unity.

i.e. $\int_0^L |\psi_n|^2 dx = 1$

$$\int_0^L \left| A\sin\frac{n\pi}{L}x \right|^2 dx = 1$$

$$\int_0^L A^2 \sin^2(\frac{n\pi x}{L})dx = 1$$

$$A^2\left[\frac{1}{2}\int_0^L dx - \frac{1}{2}\int_0^L \cos\frac{2n\pi}{L}x dx\right] = 1 \qquad [\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)]$$

Or $\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\right]_0^L = 1$

Or $\frac{A^2}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) - 0\right] = 1$

$$\Rightarrow \frac{A^2 L}{2} = 1$$

$$\Rightarrow \boxed{A = \sqrt{\frac{2}{L}}}$$

Substituting in (5) we get

$$\boxed{\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}\right)x} \quad \text{--------------------} \quad (6)$$

which are the normalized wave functions of a particle in a one-dimensional infinite potential well.

Energy Eigen values:

The energy Eigen values can be obtained by operating the wave function ψ by the energy operator (Hamiltonian operator)

$$\hat{H} = -\frac{h^2}{8\pi^2 m}\frac{d^2}{dx^2} + U$$

In the region o<x<L, V = 0 Hence U = 0

$$\therefore \hat{H} = -\frac{h^2}{8\pi^2 m}\frac{d^2}{dx^2}$$

The energy Eigen value equation is

$$\hat{H}\psi_n = E\psi_n$$

i.e. $-\frac{h^2}{8\pi^2 m}\frac{d^2}{dx^2}\psi_n = E\psi_n$ ----------------- (7)

Differentiating ψ_n in equation (6)

$$\frac{d\psi_n}{dx} = \left(\frac{n\pi}{L}\right)\sqrt{\frac{2}{L}}\cdot\cos\left(\frac{n\pi}{L}\right)x$$

and $\frac{d^2\psi_n}{dx^2} = -\left(\frac{n\pi}{L}\right)^2\sqrt{\frac{2}{L}}\cdot\sin\left(\frac{n\pi}{L}\right)x$

or $\frac{d^2\psi_n}{dx^2} = -\left(\frac{n\pi}{L}\right)^2\psi_n$

Thus equation (7) becomes $-\frac{h^2}{8\pi^2 m}\times\left[-\left(\frac{n\pi}{L}\right)^2\right]\psi_n = E\psi_n$

Or $\boxed{E_n = \frac{n^2h^2}{8mL^2}}$ ------------------------ (8)

are the energy Eigen values of the particle in an infinite potential well.

Zero-point energy:

The lowest acceptable value for n=1. Because for n=0, $\psi_n = 0$ (from equation (5)), which means that the particle is not present inside the well which is not true.

The lowest energy corresponding to n=1 is called zero-point energy and it is given by

$$E_{zero-point} = \frac{h^2}{8mL^2}$$

The lowest permitted state of energy is referred to as ground state energy. The energy state for n>1 are called exited states.

Wave functions, probability densities and Energy levels for particle in an infinite potential well:

The normalized wave functions of a particle in a one dimensional potential well of width 'L' are given by

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x \quad \text{---------------------- (1)} \qquad n = 0,1,2\ldots\ldots$$

Case1: For n=1:

This is the ground state and the particle is normally found in this state. The Eigen function corresponding to this state

$$\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}\right)x \,; \qquad \text{from (1)}$$

Here $\psi_1 = 0$ for x = 0 and x = L and is maximum for $x = \frac{L}{2}$ (fig a)

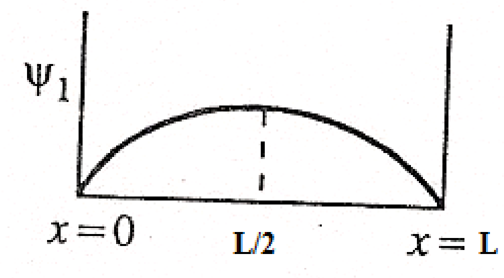


Fig (a)

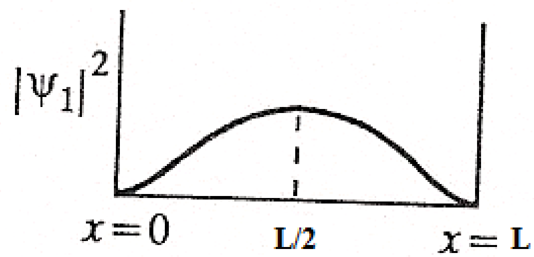


Fig (b)

A plot of $|\psi_1|^2$, the probability density verses x is shown in fig (b). It indicates the probability of finding the particle at different locations inside the well. $|\psi_1|^2 = 0$ at x = 0 and x = L and is maximum at $x = \frac{L}{2}$. This means that in the ground state the particle cannot be found at the walls of the box, and the probability of finding it is maximum at the central region.

The energy of the particle in the ground energy state is

$$E_1 = \frac{h^2}{8mL^2} = E_0, \text{ Zero-point energy}$$

<u>Case 2</u>: For n=2

This is the first exited state. The Eigen function for this state is

$$\psi_2 = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L}\right)x$$

Here $\psi_2 = 0$ for x = 0, $\frac{L}{2}$, L and maximum for $x = \frac{L}{4}$ and $\frac{3L}{4}$ (fig c)

The plot of $|\psi_1|^2$ verses x is shown in fig (d). As can be seen from the plot, the particle cannot be observed either at the walls or at the center.

The energy of the particle in this state is

$$E_2 = \frac{4h^2}{8mL^2} = 4E_0 \quad ; \qquad \because E_n = \frac{n^2h^2}{8mL^2}$$

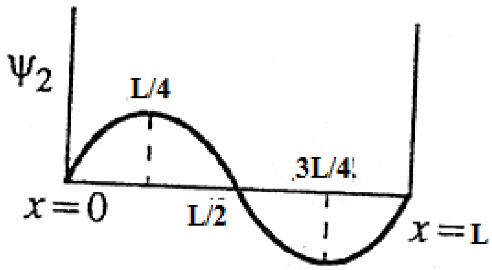


Fig (c)

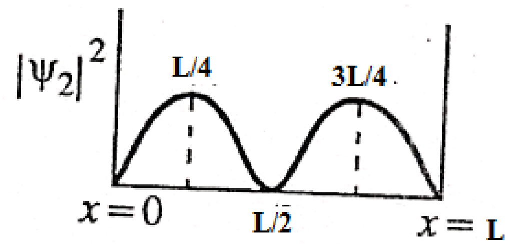


Fig (d)

Introduction to Quantum Computing

Quantum computing is a type of computation whose operations can harness the phenomena of quantum mechanics, such as superposition, interference, and entanglement. Devices that perform quantum computations are known as **quantum computers**. Developing Computing methods based on the principles of Quantum theory Quantum computers stores the information in a quantum systems in the form of Qubits. In quantum computing, electrons or photons can be considers is qubits. Instead of 0 and 1, It can choose two electronics states of an atom or two different polarization orientations of light for the two states. But as per quantum mechanics the atom can also be prepared in the state in which is said to be a coherent super imposition of both the states to represent zero and one Using quantum computing, qicker computation could be achieved and small size computer can be obtained through Quantum computing concepts

Moore's Law:

Moore's law is the observation and states that "the number of transistors in a dense integrated circuit (IC) doubles about every two years".

Moore's law is an observation and projection of a historical trend. Rather than a law of physics, it is an empirical relationship linked to gains from experience in production.

The observation is named after Gordon Moore, the co-founder of Fairchild Semiconductor and Intel. Moore's prediction has been used in the semiconductor industry to guide long-term planning and to set targets for research and development. **Moore's law helps in** reduction in quality-adjusted microprocessor prices, the increase in memory capacity (RAM and flash), the improvement of sensors, and even the number and size of pixels in digital cameras.

A silicon atom is 0.2 nanometers wide, which puts the gate length of 2 nanometers at roughly 10 silicon atoms across. At these scales, controlling the flow of electrons becomes increasingly more difficult as all kinds of quantum effects play themselves within the transistor itself. When you only have about 10 atoms distance to work with, any changes in the underlying atomic structure are going to affect the current which flows through the transistor. Since, the transistor is approaching the point where it is simply as small as we can ever make it and has it still function. The way we’ve been building and improving silicon chips is coming to its final iteration. There is also another potential pitfall for Moore's Law, and that is simple economics. The cost of shrinking transistors isn't decreasing the way it is expected.

Differences between Classical & Quantum computing

Classical Computing

1. Used by large scale, multipurpose and devices.
2. Information is stored in bits. There is discrete number of possible states. Either 0 or 1.
3. Power increases in 1:1 relationship with number of transistor
4. They have low error rates and can operate at room temperature
5. Calculations are deterministic. This means repeating the same inputs results in the same output.
6. Data processing is carried out by logic and in sequential order
7. Operations are governed by Boolean Algebra.
8. Circuit behavior is defined by Classical Physics.

Quantum Computing

1. Used by high speed, quantum mechanics-based computers.

2. Information is based on Quantum Bits. There is an infinite, continuous number of possible states. They are the result of quantum superposition.
3. Power increases in exponentially in proportion to the number of qubits
4. They have high error rates and need to be kept at ultracold
5. The calculations are probabilistic, meaning there are multiple possible outputs to the same inputs.
6. Data processing is carried out by quantum logic at parallel instances.
7. Operations are defined by linear algebra by Hilbert Space.
8. Circuit behavior is defined by Quantum Mechanics.

Concept of Qubit and its properties

Concept of Qubit

- In quantum computing, a **qubit** or **quantum bit** is a basic unit of quantum information
- A quantum system like atom or electrons can exist in states as 0 and 1 or simultaneously both as 0 and 1.
- It will not be know definitely, in which states they would be.
- But the number called probability factors associated with the corresponding state provide the probability of the atoms/electrons existence in each of these states.
- Since, they follow the quantum principles; it becomes a quantum system and called as quantum bits or Qubit.
- Similarly in case of light, a Qubit may correspond to superposed state of horizontal and vertical polarization of photons apart from two individual states.

Properties of Qubits

A Qubit can be physically implemented by the two states of an electron or horizontal and vertical polarizations of photons as $|\downarrow\rangle$ and $|\uparrow\rangle$

Superposition

A Qubit can be in a superposed state of the two states /0> and /1> **(Ket notation) .** If measurements are carried out with a qubit in superposed state then the results that we get will be probabilistic

It is represented as $/\psi> = \alpha/0> +\beta/1>$

Where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$

No Cloning Theorem

The Qubit changes its state once it is subjected to the measurement. It means, one cannot copy the information from the qubit the way we do it in classical computers, as there will be no similarity between the copy and the original. This is known as "**no cloning principle**"

Entanglement

Two Qubits can strongly correlate with each other. Changing state of one of the qubits will instantaneously change the state of the other one in predictable way. This happens even if they are separated by very long distances.

Representation of Qubits by Bloch Sphere

- The pure state space qubits (Two Level Quantum Mechanical Systems) can be visualized using an imaginary sphere called Bloch Sphere. It has a unit radius
- This Bloch sphere picture is elegant and powerful for the single qubit.
- The Arrow on the sphere represents the state of the Qubit. The north and south poles are used to represent the basis states $|0\rangle$ and $|1\rangle$ respectively. The other locations on the bloch sphere represents the superposition state i.e $|0\rangle$ and $|1\rangle$ states and represented by $\alpha\,|0\rangle + \beta\,|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Thus a Qubit can be any point on the Bloch Sphere.
- The Bloch sphere allows the state of the qubit to be represented in unit spherical co-ordinates. They are the polar angle θ and the azimuth angle ϕ. The block sphere is represented by the equation

$$|\psi\rangle = cos\ \theta/2 * |0\rangle + e^{i\phi} sin\ \theta/2 * |1\rangle$$

here $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The normalization constraint is given by

$$|\cos\theta/2|^2 + |\sin\theta/2|^2 = 1$$

- For any Gate operation, taking an initial state to the final state of the single-qubit, is equivalent to a composition of one or more simple rotations on the Bloch sphere.

Single qubit

A Single Qubit has two computational basis states |0⟩ and |1⟩. In single qubit the state of qubit will be either in |0> and |1> where

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Above basis vectors are written as column matrix in the form of identity matrix

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

which represents the single qubit state. The pictorial representation of the single qubit is as follows. $\alpha\,|0\rangle + \beta\,|1\rangle$

Two qubit

A two-qubit system has 4 computational basis states denoted as |00⟩, |01⟩, |10⟩, |11⟩.The matrix representation of two qubit state is

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

The pictorial representation of two qubit is as follows. $\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$

Transformation operation of single and two qubit system

<u>Pauli X-Gate as an example for single qubit system</u>

In Quantum Computing the Pauli X-gate is also referred as quantum NOT gate. Here qubits takes the state |0⟩ to |1⟩ and vice versa. It is analogous to the classical not gate.

The Matrix representation of Quantum Not Gate is given by

$$X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$$

A Quantum State is given by $\alpha\,|0\rangle + \beta\,|1\rangle$ and its matrix representation is given by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Hence the operation of Quantum Not Gate on quantum state is given by

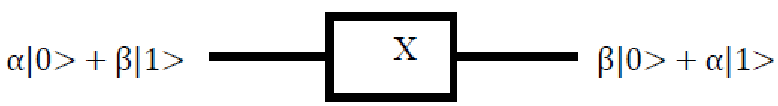
$$X\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix}$$

Thus the quantum state becomes $\alpha\,|1\rangle + \beta\,|0\rangle$. Similarly, the input $\alpha\,|1\rangle + \beta\,|0\rangle$ to the quantum not gates changes the state to $\alpha\,|0\rangle + \beta\,|1\rangle$. The quantum not gate circuit and the truth table are as shown below.

Truth table

Input	**Output**
|0>	|1>
|1>	|0>
α|0> + β|1>	β|0> + α|1>

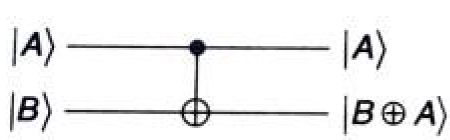
Symbol



<u>Controlled Not Gate or CNOT Gate</u>

The CNOT gate is a typical multi-qubit logic gate. The CNOT gate operates on a quantum register that consists of 2 qubits. The CNOT gate flips the second qubit (the target qubit) if and only if the first qubit (the control qubit) is |1> and the circuit is as follows.

Symbol



The Matrix representation of CNOT Gate is given by

$$U_{CN} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

The Transformation could be expressed as

$$|A, B\rangle \rightarrow |A, B \oplus A\rangle$$

Consider the operations of CNOT gate on the four inputs $|00\rangle$,$|01\rangle$,$|10\rangle$ and $|11\rangle$.

Operation of CNOT Gate for input |00⟩

$UCN|00\rangle = |00\rangle$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|0\rangle$

$$|00\rangle \rightarrow |00\rangle$$

Operation of CNOT Gate for input |01⟩

$$UCN|01\rangle = |01\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|0\rangle$. Hence no change in the state of Target qubit $|1\rangle$

$$|01\rangle \rightarrow |01\rangle$$

Operation of CNOT Gate for input |10⟩

$$UCN|10\rangle = |11\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|0\rangle$ to $|1\rangle$.

$$|10\rangle \rightarrow |11\rangle$$

Operation of CNOT Gate for input |11⟩

$$UCN|11\rangle = |10\rangle$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Here in the inputs to the CNOT Gate the control qubit is $|1\rangle$. Hence the state of Target qubit flips from $|1\rangle$ to $|0\rangle$.

$$|11\rangle \rightarrow |10\rangle$$

Truth table

Input	Output
|00>	|00>
|01>	|01>
|10>	|11>
|11>	|10>