MODULE-2
SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS.

Contents:

Introduction to second order ODE

- General form of Second order ODE with constant coefficients.
- Solution by Inverse Differential operator
  a. $\emptyset(x) = e^{ax}$
  b. $\emptyset(x) = sinax/cosax$
  c. $\emptyset(x) = e^{ax}f(x)$
- Solution of second order ODE
- Solution of Homogeneous and non-homogeneous Second order ODE
- Modelling of LCR circuit and Mass Spring System into second order ODE.
  a. Free undamped
  b. Free damped
  c. Forced Undamped
  d. Forced damped.
- Application to Oscillations of a mass spring system and L-C-R circuit.

Introduction to Numerical Solution of second order linear ODE

- Runge-Kutta method.
- Milnes Predictor Corrector method

(RBT Levels: L1, L2 & L3)

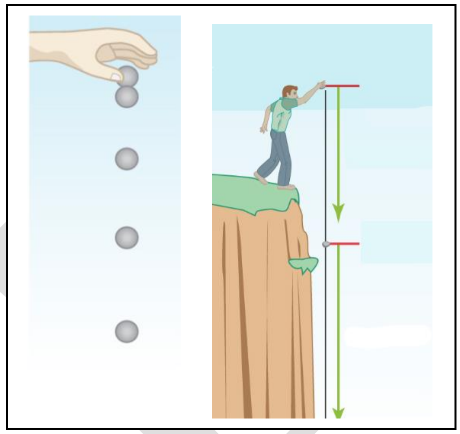
Learning Objectives:

- This course aims to provide a comprehensive understanding of modeling dynamical systems, solving ordinary differential equations through analytical and numerical methods, and utilizing MATLAB for computational applications, establishing a strong foundational knowledge in these areas.

Module Outcomes: - After Completion of this module, student will be able to

- Illustrate the fundamental concepts of Second order linear differential equations.
- Apply suitable techniques based on the knowledge acquired to solve engineering and scientific problems related to Second order linear differential equations.
- Analyze mathematical solutions of engineering and scientific problems related to Second order linear differential equations and predict their behavior in the real-world scenario.

Imagine you have **a small object**, like a **ball**, and you **drop it from a certain height**. As it falls freely under the influence of gravity, we can describe its motion using ODEs. A first-order ODE describes how the object's position changes with time. Let's call the object's position at any given time $y(t)$, where t is the time.



The first-order ODE for this scenario can be expressed as:

$$\frac{dy}{dt} = v(t)$$

Here the position of object depends on time. So, we say position variable (dependent variable) is dependent on time variable (independent variable).

How velocity relates to the falling object?

- When you initially drop the object, it starts with zero velocity. It's not moving at all at that moment.
- As the object falls under the influence of gravity, its velocity increases. This means it's speeding up as it moves downward.
- The velocity is positive (indicating downward motion) and increases as long as the object is falling freely without any other forces acting on it, like air resistance.
- If you were to measure the velocity of the object at different moments during its fall, you would find that it's getting faster and faster in the downward direction.

Let's talk about change of velocity,

Now let's focus on how velocity of object changes with respect to time.

That can be represented in terms of ODE: -

$$\frac{dv(t)}{dt} = g$$

$$\Rightarrow \frac{d\left(\frac{dy(t)}{dt}\right)}{dt} = g$$

$$\frac{d^2y(t)}{dt^2} = g$$

Which is called second order ODE, Since the order of differentiation is two.

Here the change of velocity depends on change of position (first order differentiation) with respect to time t (independent variable).

In the above second order ODE :-

- $\frac{dv(t)}{dt}$ represents the rate of change of the object's velocity with respect to time. In other words, it tells us how the velocity is changing as time progresses.
- g is the acceleration due to gravity (approximately 9.81 m/s² near the surface of the Earth).

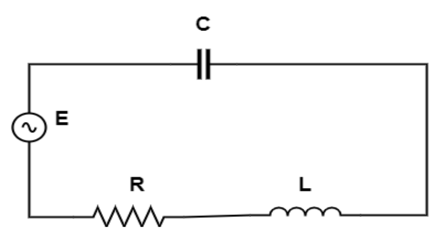
So, this second-order ODE tells us that the rate of change of velocity (how the velocity is changing over time) is equal to the constant acceleration due to gravity acting in the opposite direction of motion.

How second order ODE is connected to falling object??

- Initially, when you drop the object, it has an initial velocity of 0 (it's not moving). So, at the beginning, $\frac{dv}{dt}$ is zero.
- As time passes, the object starts to accelerate due to gravity, and this acceleration causes its velocity to increase positively.
- The object's velocity increases more and more over time due to the constant acceleration, until it reaches a point where it stops falling and begins to move upward (if there's something to stop it, like the ground or your hand).

L-C-R circuit

Consider an Electrical circuit which consists of Inductance(L), Resistance (R), Capacitance (C). Let I be the current flowing through this circuit at any time t, then the voltage across the L, C, R is given as $L\frac{dI}{dt}, \frac{q}{C}, RI$ respectively. If $E(t)$ be the external voltage applied to this series, then by Kirchhoff's law,



$$\textbf{Total Voltage} = \textbf{Voltage drop across L} + \textbf{Voltage drop across R} + \textbf{Voltage drop across C}$$

$$\boldsymbol{E(t)} = L\frac{dI}{dt} + RI + \frac{q}{C}$$

But current is the rate of change of charge with respect to time. $\boldsymbol{I = \frac{dq}{dt}}$

Therefore, the above equation becomes,

$$\boldsymbol{E(t)} = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

Divide the equation by L, we get

$$\boldsymbol{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E(t)}{L}} \qquad (1)$$

If the External voltage is zero $(E(t) = 0)$, then the circuit is said to be free.

Then the equation (1) becomes,

$$\boldsymbol{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0} \qquad (2)$$

The Equation (1) and (2) is called **second order ordinary linear differential equation** with **constant coefficient**. Because the degree of highest derivative is one. The equation obtained when there is a External voltage $E(t)$ is called non-homogeneous order ordinary linear differential equation and the equation obtained when there is a no External voltage $(E(t) = 0)$is called homogeneous second order ordinary linear differential equation.

General form of linear Second order ODE:-

The Second order ODE is said to be linear if it is in the form of

$$\boldsymbol{\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)}$$

Or

$$\boldsymbol{y'' + p(x)y' + q(x)y = f(x)}$$

and nonlinear if it is cannot written in this form. Where p, q, r are the function of x.
If the first term is, say $f(x)y''$, then divide by $f(x)$ to get standard form (1). If $r(x) = 0$ then

$$\boldsymbol{y'' + p(x)y' + q(x)y = 0}$$

is called homogeneous. If $r(x) \neq 0$, then (1) is called non-homogeneous.

Second order ODE with constant coefficients.

We now proceed to study those second order linear equations which have constant coefficients. The general form of such an equation is:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \longrightarrow \quad (3)$$

Where a,b,c are some constants. The homogeneous form of (3) is the case when $r(x) = 0$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \longrightarrow \quad (4)$$

Solution of Second order ordinary linear Differential equation (In L-C-R Circuit)

The solution of second-order linear ODE describes the behavior of the circuit in response to changes in voltage or current. The General solution is the sum of the complementary solution and particular solution.

The general solution of the ODE represents a family of solutions that includes all possible solutions to the differential equation. It typically contains arbitrary constants that need to be determined based on initial conditions or other constraints specific to the problem. (In the context of an LCR circuit, the general solution typically includes both the transient behavior and the steady-state behavior of the circuit)

The complementary solution represents the solution to the homogeneous part of the differential equation, which is the part where the right-hand side (RHS) is zero (i.e., E(t)=0). In the context of the LCR circuit, it describes how the circuit behaves in the absence of any external voltage source.(The complementary solution (also known as the natural response or transient response) of an LCR circuit describes the behavior of the circuit when it is subjected to initial conditions (e.g., initial voltages or currents) and then left to evolve on its own without any external sources (e.g., voltage or current sources).

The particular solution represents the solution to the non-homogeneous part of the differential equation, which is the part where the right-hand side (RHS) is not zero (i.e., E(t) is a non-zero function of time). It describes how the circuit responds to the external voltage source. The particular solution depends on the specific form of V(t).(The particular integral represents the forced response of the circuit, which is caused by external sources such as voltage or current sources. It describes how the circuit responds to external inputs.

The **general solution** (Complete solution) of (3) is $y = y_c + y_p$, where y_c is called complementary function and y_p is called particular solution (integral).

To find the general solution of (3), it is first necessary to solve (4). The general solution of (4) is called the complementary function and will always contain two arbitrary constants. We will denote this solution by y_c

The technique for finding the complementary function.:-

To find the complementary function we must make use of the following property.

If $y_1(x)$ and $y_2(x)$ are any two (linearly independent) solutions of a linear, homogeneous second order differential equation then the complementary function is $y_c = C_1y_1(x) + c_2y_2(x)$

Where $c_1$and c_2 are constants. To get the value of $y_1(x)$ and $y_2(x)$, we look at the **characteristic equation(Auxiliary Equation)**, obtained by replacing, $\frac{d}{dx}$ by D and $\frac{d^2}{dx^2}$ by D^2,(where D is called Differential operator) in ODE we get

$$(aD^2 + bD + c)y = 0 \quad \longrightarrow \quad (5)$$

Where a, b, c are some constants. We form the 'Auxiliary equation', $f(m) = 0$ and solve the same to obtain the roots $m_1$and m_2.

We distinguish between 3 cases: the case when the roots of the characteristic equation are distinct and real, complex, or equal.

Case 1: Real and distinct roots m_1 and m_2.

Then the solution of differential equation of the form,

$$\text{y} = y_c = c_1e^{m_1x} + c_2e^{m_2x}, (m_1 \neq m_2)$$

Example:- Find the solution of

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Solution:- The given Second order ODE can be put in the form of $f(D)y = 0$

We get, $(D^2 + 5D + 6)y = 0$

Auxiliary equation is, $f(m) = 0$

$$(m^2 + 5m + 6) = 0$$

$$m_1 = -3 \text{ and } m_2 = -2$$

Therefore, the general solution is $y = y_c = c_1e^{-3x} + c_2e^{-2x}$

Example:- Solve the initial value problem.

$$y'' + y' - 2y = 0; \text{Given } y(0) = 4; y'(0) = -5$$

The characteristic equation $m^2 + m - 2 = 0$

The roots are $m_1 = 1$ and $m_2 = -2$

$$y = y_c = c_1 e^{x} + c_2 e^{-2x} \longrightarrow \quad (a)$$

By Differentiation we obtain $y' = y_c' = c_2 e^{2x} + 2(c_1 + c_2 x)e^{2x}$

From this and initial conditions it is follows that $y(0) = c_1 + c_2 = 4$ and

$$y'(0) = c_1 + 2c_1 = -5$$

By solving above equation we get $c_1 = 1$ and $c_2 = 3$

Hence, $y = e^{x} + 3e^{-2x}$

Case 2: Complex roots $\alpha \pm i\beta$.

Then the solution of differential equation of the form,

$$y = y_c = e^{\alpha x}[c_1 cos(\beta x) + c_2 sin(\beta x)]$$

Example:- Find the solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0$$

Solution:- The given Second order ODE can be put in the form of $f(D)y = 0$

We get, $(D^2 + 4D + 9)y = 0$

Auxiliary equation is, $f(m) = 0$

$$(m^2 + 4m + 9) = 0$$

$$m_1 = -2 + \sqrt{5}i \text{ and } m_2 == -2 - \sqrt{5}i$$

Therefore, the general solution is, $y = y_c = e^{-2x}\left[c_1 cos\left(x\sqrt{5}\right) + c_2 sin\left(x\sqrt{5}\right)\right]$

Example:- Solve the initial value problem.

$y'' - 0.2y' + 4.01y = 0$; Given $y(0) = 0; y'(0) = 2$

The characteristic equation $m^2 - 0.2m + 4.01 = 0$

The roots are $m_1 = -0.1 + 2i$ and $m_2 = -0.1 - 2i$

$y = y_c = e^{-0.1x}(c_1 cos2x + c_2 sin2x)$

From the initial condition $y(0) = A = 0$. There remains $y = Be^{-0.1x} sin2x$

$y' = B(-0.1e^{-0.1x} sin2x + 2e^{-0.1x} cos2x)$

By substituting second initial condition $y'(0) = 2B = 2$ Hence B=1

$$y = e^{-0.1x} sin2x$$

Case 3: Equal roots $m_1 = m_2 = m$:-

Then the solution of differential equations of the form

$$\boldsymbol{y = y_c = (c_1 + c_2 x)e^{mx}}$$

Example:- Find the solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

Solution:- The given Second order ODE can be put in the form of $f(D)y = 0$

We get, $(D^2 + 4D + 4)y = 0$

Auxiliary equation is, $f(m) = 0$

$$(m^2 + 4m + 4) = 0$$

$$(m + 2)^2 = 0$$

$$m_1 = -2 \text{ and } m_2 = -2$$

Therefore, the general solution is, $y = y_c = e^{-2x}[c_1 + c_2x]$

Example: - Solve the initial value problem.

$$y'' - 4y' + 4y = 0; \text{ Given } y(0) = 3; y'(0) = 1$$

The characteristic equation $m^2 - 4m + 4 = 0$

$$(m - 2)^2 = 0$$

$$m = 2,2$$

$$y' = y_c = (c_1 + c_2x)e^{(2x)} \longrightarrow \text{(a)}$$

By Differentiation we obtain $y' = y_c' = c_2e^{2x} + 2(c_1 + c_2x)e^{2x}$

From this and initial conditions it is follows that $y(0) = c_1 = 3$ and $y'(0) = c_2 + 2c_1 = 1$

Solution of non-homogeneous equations: -

As we have discussed earlier the general solution of the non-homogeneous equation is of the form

$$y = y_c + y_p$$

The solution of a nonhomogeneous differential equation can be found using the following steps:

1. Find the general solution of the associated homogeneous equation.
2. Find a particular solution of the nonhomogeneous equation.
3. The general solution of the nonhomogeneous equation is the sum of the general solution of the homogeneous equation and the particular solution.

Inverse Differential operator and particular integral (particular solution) :-

The inverse differential operator in a second order ODE is an operator that reverses the action of the second order derivative operator. In other words, if D is the second order derivative operator, then the inverse differential operator, denoted by D^{-1} or $\frac{1}{D}$, is an operator that $DD^{-1}\big(f(x)\big) = f(x)$. The operator $\frac{1}{D}$ that stands for the integral is called an inverse differential operator. The terms $\frac{1}{D}, \frac{1}{D^2}$ ect., Stands for successive integration.

The inverse differential operator can be used to find the particular solution of a nonhomogeneous second order ODE. The particular solution is the solution that is not a solution of the associated homogeneous equation.

Evaluation of particular integral

The particular integral of the given DE can be evaluated as $y_p = \frac{\emptyset(x)}{f(D)}$, where Q(x) is a function in the variable of x and it can be determined as follows,

- If $\emptyset(x) = e^{ax}$, then $y_p = \frac{e^{ax}}{f(D)}$

$$.\Rightarrow y_p = \frac{e^{ax}}{f(a)}, where\ f(a) \neq 0$$

and $y_p = \frac{e^{ax}x}{(D-a)^k Q(D)} = \frac{x^k e^{ax}}{k! Q(a)}$ where $f(a) \neq 0$

- If $\emptyset(x) = cosax$ (or) $sinax$ then y_p becomes $y_p = \frac{cosax}{f(D)}$ or $y_p = \frac{sinax}{f(D)}$

  To solve the above replace $D^2 = -a^2$ and solve the same for the denominator not equal to zero and if the denominator becomes zero, we have $y_p = \frac{cosax}{D^2+a^2} = \frac{xsinax}{2a}$ and

$$y_p = \frac{sinax}{D^2 + a^2} = \frac{-xcosax}{2a}$$

  After Replacing D^2 by $-a^2$, $f(D)$ transforms into the form $\alpha D \pm \beta$ (α and β are constants) in which case we multiply both the numerator and denominator by $\alpha D \mp \beta$ for proceeding further in obtaining the PI.

- If $\emptyset(x) = e^{ax} f(x)$, then $y_p = \frac{e^{ax} f(x)}{f(D+a)}$

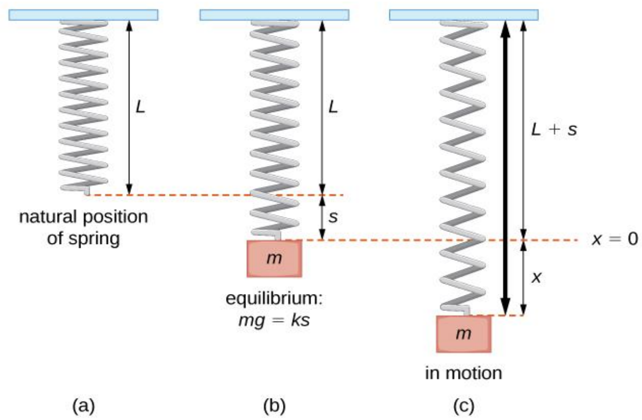
  If e^{ax} is multiplied with some function of x, we need to first shift D to D+a. Then compute PI follow type B.

Mass Spring Problem:

Consider a mass suspended from a spring attached to a rigid support. (This is commonly called a spring-mass system.) Gravity is pulling the mass downward (Gravitational force $F = mg$) and the restoring force of the spring is pulling the mass upward($F = ks$) where k is the spring constant and

$$F = m.g = ks \qquad (5)$$

As shown in Figure (b), when these two forces are equal, the mass is said to be at the equilibrium position. If the mass is displaced from equilibrium, it oscillates up and down. This behavior can be modeled as follows,



When the particle is displaced from x = 0, the force F tends to draw the particle back towards x = 0. When we pull the spring from $x = 0$ (say at a displacement x(t) >0), and releasing, it oscillates up and down. Then according to Newton's second law of motion,

$$F_{net} = m.a \qquad (6)$$

For free undamped system (No force opposes the motion and undamped oscillation of object):

The net force acting on the object is Gravitational force and according to Hooke's law, the restoring force of the spring is proportional to the displacement and acts in the opposite direction from the displacement i.e, $F = -k(x + s)$,

Therefore from (6)

$$F_{net} = m.x''$$

$$m.x'' = mg - k(s + x)$$

From (5) $m.g = ks$

$$\therefore\ \boldsymbol{m.x'' + kx = 0} \qquad (7)$$

Or

$$\boldsymbol{m.x'' + \omega^2 x = 0} \ \ \textbf{where } \boldsymbol{\omega = \sqrt{\frac{k}{m}}}$$ This represents Mechanical system

The general solution of Above equation will be $\mathbf{x = A\cos(\omega t) + B\sin(\omega t)}$ (8)

Which can also be written as $x = A\cos(\omega t + B)$ $[\because A\cos x + B\sin x = \sqrt{A^2+B^2}\cos(x \pm B); \tan B = \frac{\sin B}{\cos B}]$. Equation 8 represents Oscillatory vibrations of variable x with amplitude A and period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k}{m}}$.

Note: In an **electrical circuit**, a free undamped situation is demonstrated in an LC circuit. The governing equation for this scenario is $L\frac{dI}{dt} + \frac{q}{C} = 0$,

$$\Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

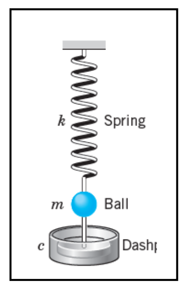
$$\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0 \text{ where } \omega = \frac{1}{LC}$$

The period **T** of the oscillatory current is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\mathbf{LC}}$ representing a cycle of $2\pi\sqrt{\mathbf{LC}}$ in the oscillations.

For damped system (a resistance force that opposes the motion or oscillation of an object)

Let a dashpot (speed breaker which reduces the oscillation of spring and bring it to equilibrium position) is attached to the mass-spring system. The damping force is directly proportional to the velocity of the object. Therefore, $F = -cx'$.



The net force acting on the object is Gravitational force and according to Hooke's law, the restoring force of the spring is proportional to the displacement and acts in the opposite direction from the displacement law $F = -k(x+s)$ and damping force.

From (6)

$$F_{net} = m.x'' = mg - k(s+x) - cx'$$

$$m.g = ks$$

$$m.x'' + cx' + kx = 0 \quad (8)$$

The Equation (7) and (8) is called **second order linear differential equation** because the degree of highest derivative is one.

$$x'' + \frac{c}{m}x' + \frac{c}{m}x = 0$$

Solution of this quadratic equation will be $m = \frac{-c \pm \sqrt{c^2-4km}}{2m}$

Equivalent Electrical for **damped system**,

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

Solution of this equation Is,

$$m = \frac{-R \pm \sqrt{R^2 - \frac{4C}{L}}}{2C}$$

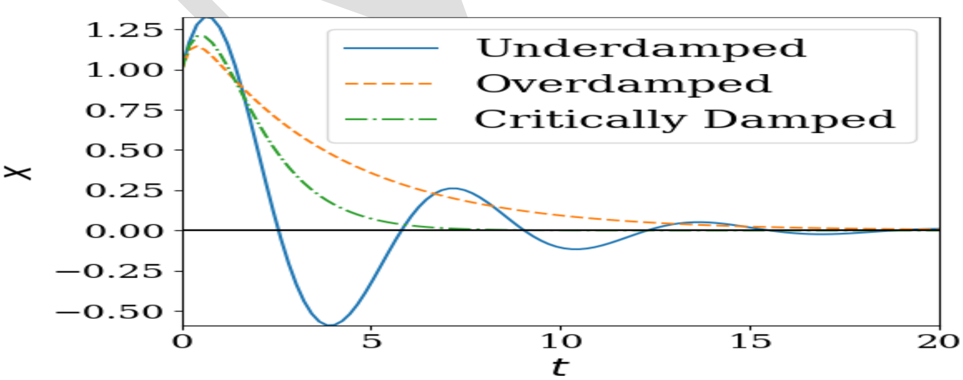
Interpretation of solution:

Note: The auxiliary equation of (2) is $Lm^2 + Rm + \frac{1}{C} = 0$

Solution to this quadratic equation is $m = \frac{-R\pm\sqrt{R^2-\frac{4L}{c}}}{2L}$

Then we say the circuit is,

- Over damped if $R^2 - \frac{4L}{C} > 0$, then the solution is of type **case 1. (Solution is always positive and motion (behavior of current) is non oscillatory)**
- Critically damped, if $R^2 - \frac{4L}{C} = 0$, then the solution is of type **case 3. (Solution is always positive and motion (behavior of current is non oscillatory)**
- Under damped if $R^2 - \frac{4L}{C} < 0$, then the solution is of type **case 2. (Motion (behavior of current is oscillatory due to trigonometric function in the solution)**



Forced Oscillation (without damping):Forced Undamped

The forced undamped response in a mass-spring system refers to the behavior of the system when subjected to an external force, but without any damping effect. The equation of motion for a forced undamped mass-spring system is given by:
If a periodic force $f(t)$ is applied then equation 7 becomes

$$\boldsymbol{m.x'' + kx = f(t)}$$

Consider a forced undamped response in the context of an LCR circuit, which consists of an inductor (L), capacitor (C), and resistor (R). The governing equation for the circuit subjected to an external force is derived from Kirchhoff's voltage law and is given by:

$$\boldsymbol{L\frac{d^2q}{dt^2} + \frac{q}{c} = f(t)}$$

Forced Oscillation (with damping) Forced damped:

Consider a mass-spring system with damping, subject to an external force $f(t)$. The equation of motion for this system is given by:

$$x'' + \frac{c}{m}x' + \frac{c}{m}x = \frac{f(t)}{m}$$

The forced damped response in an LCR circuit describes the behavior of the circuit when subjected to an external current $E(t)$, accounting for damping effects. The governing equation for a forced damped LCR circuit is derived from Kirchhoff's voltage law and is given by:

$$\boldsymbol{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E(t)}{L}}$$

Oscillatory systems

	Mass Spring	LCR
Free Undamped	$m.x'' + kx = 0$	$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$
Free Damped	$x'' + \frac{c}{m}x' + \frac{c}{m}x = 0$	$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$
Forced Undamped	$m.x'' + kx = f(t)$	$L\frac{d^2q}{dt^2} + \frac{q}{C} = f(t)$
Forced Damped	$x'' + \frac{c}{m}x' + \frac{c}{m}x = \frac{f(t)}{m}$	$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E(t)}{L}$

APPLICATIONS OF MASS SPRING SYSTEM:

- Musical (Instruments): Musical instruments like pianos and guitars use mass spring systems in their sound production For example, piano strings are attached to a soundboard (mass spring system) that resonates to produce music.
- Shock absorption in Sports Equipment : Sports equipment, such as shoes, athletic footwear, and even sports courts, often use mass spring systems to provide shock absorption and improve performance while minimizing the risk of injury.

- Mechanical Testing:
  Mass spring systems are used in mechanical testing equipment to apply controlled forces and measure the response of materials They are important for material testing and quality control
- Energy Storage :
  Mechanical energy can be stored and released using mass spring systems This concept is applied in various energy storage systems, such as flywheels, where the rotational motion of a mass spring system stores energy for later use.
- Aerospace
  In aerospace engineering, mass spring systems are utilized for damping and vibration control in aircraft and spacecraft They help reduce oscillations and vibrations, enhancing the safety and stability of aerospace systems.

- Human Body Modeling
  Mass spring systems are employed in biomechanics to model and analyze the dynamics of the human body They are used to study how the body responds to external forces and impacts, which is vital in fields like sports science and vehicle safety.
- Shock Absorbers in Buildings
  Mass spring damper systems are used in the design of structural and architectural components to absorb and dissipate energy from environmental forces, such as wind or seismic activity.

These applications highlight the versatility of mass spring systems in controlling motion, vibration, and oscillation, making them essential in many fields where dynamic behavior needs to be managed or leveraged for various purposes.

APPLICATIONS OF RLC-CIRCUIT

RLC circuits are widely used in,

1. Filter design where they can act as low pass, high pass, band pass, or band stop filters These filters are used in audio equipment, communications systems, and signal processing to allow or block certain frequency components
2. Tuned Circuits such as radio receivers and television sets They are used to select a specific frequency from a mixture of signals and reject others These circuits help in tuning to different radio stations or TV channels
3. Signal Processing Signal conditioning, and noise filtering.
4. MRI Machines To generate and detect radiofrequency signals

These are just a few applications of RLC circuit.

Problems.

1. Solve $(D^2+4D+3)y=e^{-x}$

Ans: Given DE is,

$(D^2+4D+3)y=e^{-x}$

$\Rightarrow f(D)y=e^{-x}$

Where $f(D)=D^2+4D+3$

$\therefore$ the auxiliary equation is,

$f(m)=0$

$m^2+4m+3=0$

$m^3+m+3m+3=0$

$m(m+1)+3(m+1)=0$

$m+1=0;\ m+3=0$

$m=-1,-3$

$\therefore y_c=c_1e^{-x}+c_2e^{-3x}$

$\therefore y_p=\frac{e^{-x}}{f(D)}$

$$y_p = \frac{e^{-x}}{D^2 + 4D + 3}$$
$$y_p = \frac{e^{-x}}{(D+1)(D+3)}$$
$$y_p = \frac{x^1}{1!}\frac{e^{-x}}{D+3}$$
$$y_p = \frac{x^1}{1!}\frac{e^{-x}}{(-1+3)}$$
$$y_p = \frac{xe^{-x}}{2}$$
$$y = y_c + y_p$$
$$y = c_1 e^- x + c_2 e^{3x} + \frac{xe^{-x}}{2}$$

2. Solve $(D^2 - 4D + 13)y = cos2x$

Ans: The given DE is $(D^2 - 4D + 13)y = cos2x$
$\Rightarrow f(D)y = cos2x$
Where $f(D) = D^2 - 4D + 13$
$\therefore The\ auxiliary\ equation\ is$
$f(m) = 0$
$\Rightarrow m^2 - 4m + 13 = 0$

$m = 2 \pm 3i$
$\therefore y_c = (c_1 cos3x + c_2 sin3x)$
$$y_p = \frac{cos2x}{f(D)}$$
$$= \frac{cos2x}{D^2 - 4D + 13}$$
$$y_p = \frac{cos2x}{-4 - 4D + 13} (\because D^2 = -2^2)$$
$$\Rightarrow y_p = \frac{cos2x}{9 - 4D}$$
$$= \frac{(9 + 4D)cos2x}{(9 - 4D)(9 + 4D)}$$
$$= \frac{(9 + 4D)cos2x}{81 - 16D^2}$$
$$= \frac{(9 + 4D)cos2x}{81 - 16(-4)} (\because D^2 = -2^2)$$
$$= \frac{(9 + 4D)cos2x}{145}$$
$$y_p = \frac{9cos2x - 8sin2x}{145}$$
$$\therefore y = y_c + y_p$$
$$y = (c_1 cos3x + c_2 sin3x) + \frac{9cos2x - 8sin2x}{145}$$

3. Solve $\left(\frac{d^2y}{dx^2} - 4y\right) = cosh(2x-1) + 3^x$

Given DE is $\left(\frac{d^2y}{dx^2} - -4y\right) = cosh(2x-1) + 3^x$

$\Rightarrow f(D)y = cosh(2x-1) + 3^x$ where $D = \frac{d}{dx}$

$f(D^2-4)y = 0$

$f(D) = D^2 - 4$

$\therefore$ *the auxiliary equation is,*

$f(m) = 0$

$\Rightarrow m^2 - 4 = 0$

$\Rightarrow m^2 - 2^2 = 0$

$\Rightarrow\ m = -2, 2$

$\therefore y_c = c_1e^{-2x} + c_2e^{2x}$

$y_p = \frac{cosh(2x-1)+3^x}{f(D)}$

$= \frac{cosh(2x-1)+3^x}{D^2-4}$

$$y_p = \frac{\frac{e^{2x-1}+e^{-(2x-1)}}{2} + e^{(log_e 3)x}}{D^2-4}$$

$$\Rightarrow y_p = \frac{e^{2x-1}}{2(D^2-4)} + \frac{e^{-2x+1}}{2(D^2-4)} + \frac{e^{(log\,3)x}}{D^2-4}$$

$$\Rightarrow y_p = \frac{e^{2x-1}}{2(D-2)(D+2)} + \frac{e^{-2x+1}}{2(D-2)(D+2)} + \frac{e^{(log\,3)x}}{D^2-4}$$

$$\Rightarrow y_p = \frac{xe^{2x-1}}{2.1!.(D+2)} + \frac{xe^{-2x+1}}{2.1!.(D-2)} + \frac{e^{(log\,3)x}}{D^2-4}$$

$$\Rightarrow y_p = \frac{xe^{2x-1}}{2.4} + \frac{xe^{-2x+1}}{2.-4} + \frac{e^{(log\,3)x}}{(log\,3)^2-4}$$

$$= \frac{x}{4}\left(\frac{e^{2x-1}-e^{-(2x-1)}}{2}\right) + \frac{3^x}{(log\,3)^2-4}$$

$$= \frac{x}{4}sinh(2x-1) + \frac{3^x}{(log3)^2-4}$$

4. Solve $(D^2-6D+9)y = 6e^{3x} + 7e^{-2x} - \log 2$

Ans: The give DE is,

$(D^2-6D+9)y = 6e^{3x} + 7e^{-2x} - \log 2$

$\Rightarrow f(D)y = 6e^{3x} + 7e^{-2x} - \log 2$

Where $f(D) = D^2 - 6D + 9$

$\therefore$ the auxiliary Equation is,

$f(m) = 0$

$\Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow\ m = 3,3$

$\therefore y_c = (c_1 + c_2 x)e^{3x}$

$y_p = \dfrac{6e^3x + 7e^{-2x} - \log 2}{f(D)}$

$= \dfrac{6e^{3x}}{(D-3)^2} + \dfrac{7e^{-2x}}{(D-3)^2} - \dfrac{\log 2\, e^{0x}}{(D-3)^2}$

$= \dfrac{6x^2}{2!}\dfrac{e^{3x}}{1} + \dfrac{7e^{-2x}}{25} - \dfrac{\log 2\, e^{0x}}{9}$

$y_p = 3x^2e^{3x} + \dfrac{7e^{-3x}}{25} - \dfrac{\log 2}{9}$

$\therefore y = y_c + y_p$

$y = (c_1 + c_2 x)e^{3x} + 3x^2e^{3x} + \dfrac{7e^{-3x}}{25} - \dfrac{\log 2}{9}$

5. Find the charge on the capacitor in an RLC series circuit where L=5/3 H, R=10Ω, C=1/30 F, and E(t)=300 V. Assume the initial charge on the capacitor is 0 C and the initial current is 9 A. What happens to the charge on the capacitor over time?

Solution: We have,

$$L\frac{dq^2}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$\frac{5dq^2}{3dt^2} + 10\frac{dq}{dt} + 30q = 300$$

$$\frac{dq^2}{dt^2} + 6\frac{dq}{dt} + 18q = 180$$

The general solution is $e^{-3t}(c_1 cos3t + c_2 sin3t)$

Assume a particular solution of the form qp=A, where A is a constant. Using the method of undetermined coefficients, we find A=10. So,

$$q(t) = e^{-3t}(c_1 cos3t + c_2 sin3t) + 10.$$

Applying the initial conditions q(0)=0 and i(0)= $\frac{dq}{dt}$(0)=9, we find c_1=−10 and c_2=−7.

So the charge on the capacitor is

$$q(t)=-10e^{-3t}\cos(3t)-7e^{-3t}\sin(3t)+10.$$

Looking closely at this function, we see the first two terms will decay over time (as a result of the negative exponent in the exponential function). Therefore, the capacitor eventually approaches a steady-state charge of 10C.

Practice Problems:

1. $y''+4y'+4y=0$ given that $y=0, y'=-1$ at $x=1$.
2. Solve $(D^2+4D+3)y=e^{-x}$
3. $(D-2)^2 y=8(e2x+\sin 2x)$
4. Solve $(D^2-4D+13)y=cos2x$
5. $(D^2-3D+2)y=2e^x cos\frac{x}{2}$
6. $y''+y'-2y=0; given y(0)=4; y'(0)=-5$
7. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}+cos(2x)+4$
8. $y''-3y'+2y=2sinx\ cosx$
9. $Solve\ y''-0.2y'+4.01y=0; Giveny(0)=0; y'(0)=2$
10. $Solve(D^2-6D+9)y=6e^{3x}+7e^{-2x}-log2$
11. Determine the charge on the capacitor at any time t > 0 in circuit series having an emf E(t) = 100 sin60t, a resistor of 2Ω, and inductor of 0.1H and capacitor of $\frac{1}{260}$F, if the initial current and charge on the capacitor are both zero.
12. An electric circuit consists of an inductance of 0.1H a resistance of 20Ω and a condenser of 25 μF. Find the charge q and current i at any time t, given that q(0) = 0.05 and i(0)=0.
13. Assume an object weighing 2 lb stretches a spring 6 in. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 16 ft/sec. What is the period of the motion?
14. A 16-lb mass is attached to a 10-ft spring. When the mass comes to rest in the equilibrium position, the spring measures 15 ft 4 in. The system is immersed in a medium that imparts a damping force equal to 5252 times the instantaneous velocity of the mass. Find the equation of motion if the mass is pushed upward from the equilibrium position with an initial upward velocity of 5 ft/sec. What is the position of the mass after 10 sec? Its velocity?
15. A body weighing 10kg is hung from a spring. A pull of 20kg wt will stretch the spring to 10cm. The body is pulled down to 20cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation.
16. An 8 lb weight is placed at one end of a spring suspended from the ceiling. The weight is raised to 5 inches above the equilibrium position and left free. Assuming the spring constant 12 lb/ft, find the equation of motion, displacement function x(t), amplitude, period, frequency and maximum velocity.

17. At $t = 0$ a current of 2 amperes flows in an LCR circuit with resistance R=40ohms, inductance L=0.2 henrys, and capacitance $C = 10^{-5}$farads. Find the current flowing in the circuit at t>0 if the initial charge on the capacitor is 1 coulomb. Assume that E(t)=0 for t>0.

18. A circuit consists of an inductance of 0.05 henrys, a resistance of 5 ohms and a condenser of capacitance 4 × 10−4 farad. If Q = I = 0 when t = 0, find Q(t) and I (t) when (a) there is a constant emf of 110 volts (b) there is an alternating emf s 100t . (c) Find the steady-state solution in (b).

Numerical Solution of second order ODE:- Most ordinary differential equations arising in real-world applications cannot be solved exactly. These ODE can be analyzed qualitatively. However, qualitative analysis may not be able to give accurate answers. A numerical method can be used to get an accurate approximate solution to a differential equation. Two widely used numerical methods for solving second-order ODEs are the Runge-Kutta method and the Milne's Predictor-corrector method.

Consider a Second order ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x) \longrightarrow (1)$$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$ then, (1) => $\frac{dz}{dz} = g(x, y, z)$ to the initial conditions

$$y(x_0) = y_0;\ y'(x_0) = y_0';z(x_0) = z_0$$

a) **R.K Method**: - The Runge-Kutta method is a popular numerical technique for solving initial value problems involving ordinary differential equations. The Runge-Kutta method involves iteratively updating the solution from an initial point to approximate the solution at successive points in the domain. It is based on weighted averages of function evaluations at multiple intermediate points within each step. We can find the solution at $x = x_1$ by following,

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$ where,

$k_1 = hf(x_0, y_0, z_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$l_1 = hg(x_0, y_0, z_0)$

$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$l_4 = hg(x_0 + h, y_0 + K_3, z_0 + l_3)$

1) Using RK method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$ correct to 4 decimals using initial conditions $y(0) = 1, y' = 0$.

Sly:- Given $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2 - (1)$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$(1) \Rightarrow \frac{dz}{dx} = xz^2 - y^2 = g(x,y,z)$

and $y'(0) = 0, y_{(0)} = 1$

$\Rightarrow x_0 = 0, y_0 = 1\ y'_0 = z_0 = 0, h = 0.2$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.2)(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.2)g(0,1,0)$

$= (0.2)(-1) \Rightarrow l_1 = -0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.2)f(0.1,1,-0.1)$

$= (0.2)(-0.1)$

$= -0.02$

$\therefore l_2 = hg(x_1, y_1, z_1)$

$= (0.2)(0.1,1,-0.1)$

$= (0.2)(-0.999)$

$= -0.2$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.2) + (0.1,0.99,-0.1)$

$= (0.2)(-0.1)$

$= -0.02$

$\therefore l_3 = (0.2)g(0.1,0.99,-0.1)$

$= (0.2)(-0.989)$

$l_3 = -0.1978$

$k_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= (0.2) + (0.2,0.98,-0.1978)$

$= (0.2)(-0.1978)$

$K_4 = -0.03956$

$l_4 = hg(x_0 + h, y_0 + K_3, z_0 + l_3)$

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$1 + \frac{1}{6}[0 - 0.04 - 0.04 - 0.03956]$$

$y(0.2) \cong 0.98$

b) **Milne's Predictor Corrector Method**:- Milne's method is a numerical method for solving ordinary differential equations. It is a predictor-corrector method, which means that it uses two steps to approximate the solution: a predictor step and a corrector step. In the predictor step, a linear approximation to the solution is made. This approximation is then used to compute the solution at the next time step. In the corrector step, the actual solution is computed at the next time step and

compared to the predicted solution. The difference between the two solutions is then used to improve the accuracy of the approximation. We can find the solution at $x = x_1$ by following,

$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3);\ z_4^p = z_0 + \frac{4h}{3}(2g_1 - g_2 + 2g_3)$

$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p);\quad z_4^c = z_2 + \frac{h}{3}(g_2 + gz_3 + g_4^p)$

Example: Apply the milne's predictor corrector method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$

X	0	0.2	0.4	0.6
Y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

Given $y'' = 1 - 2yy' - (1)$

Let $y' = z = f(x, y, z)$

$\therefore$ (1) $\Rightarrow \frac{dz}{dx} = 1 - 2yz = g(x, y, z)$

$x_0 = 0 \quad y_0 = 0 \quad y_0' = z_0 = 0$
$x_1 = 0.2 \quad y_1 = 0.02 \quad y_1' = z_1 = 0.1996$
$x_2 = 0.4 \quad y_2 = 0.0795 \quad y_2' = z_2 = 0.3972$
$x_3 = 0.6 \quad y_3 = 0.1762 \quad y_3' = z_3 = 0.5689$

$f_0 = f(x_0, y_0, z_0) = 0;\ q_0 = 1 - 2y_0z_0 = 1;$

$f_1 = f(x_1, y_1, z_1) = 0.1996;\ q_1 = 1 - 2y_1z_1 = 0.9920$

$f_2 = f(x_2, y_2, z_2) = 0.3972;\ q_2 = 1 - 2y_2z_2 = 0.93684$

$f_3 = f(x_3, y_3, z_3) = 0.5689;\ q_3 = 1 - 2y_3z_3 = 0.7995$

$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$$= 0 + \frac{4 \times 0.2}{3}[2 \times 0.1996 - 0.3972 + 2 \times 0.5689]$$

$$\therefore y_4^{(P)} = 0.3039$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}[291 - 92 + 293]$$

$$= 0 + \frac{4 \times 0.2}{3}[2.984 - 0.93684 + 1.599]$$

$$z_4^{(P)} == 0.7056$$

$$f_4^{(P)} = 0.7056$$

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[f2 + 4f3 + f_4^{(P)}\right]$$

$$= 0.0795 + \frac{0.2}{3}[0.3972 + 4 \times 0.5689 + 0.7056]$$

$$= 0.3047$$

Practice Problems:

1) Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$ for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 1$

   $y'(0) = 0$ using R-K Method of 4th Order.

   $Ans: y(0.1) = 1.0053$

2) Solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 3$; $y'(0) = 0$ using R-K Method of 4th Order.

   $Ans: y(0.1) = 2.9849$

3) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y$ =0 for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 2$; $y'(0) = 1$ using R-K Method of 4th Order.

4) Solve $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = y$ for $x = 0.2$ correct to 4 decimal places using the conditions $y(0) = 1$; $y'(0) = 0$ using R-K Method of 4th Order.

Ans: $y(0.2) = 0.9802; y' = -0.196$

5) Use R.K Method solve $y'' - xy + 4y = 0$; $y(0) = 3$; $y'(0) = 0$ at $x = 0.1$

Ans:2.9399

6) Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 0$ given that $y(0) = 1$; $y'(0) = 0$; for y(0.1) using R-K Method.

Ans: 1.0053

7) Solve $y' = x + z, z' = x - y$ for $x = 0.1$ given that $y = 0, z = 1\ at\ x = 0$ using R-K Method

Ans:0.1050

8) Apply the milne's predictor corrector method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$

X	0	0.2	0.4	0.6
Y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

$y_4^{(p)} = 0.3039$; $z_4^{(p)} = 0.7050$; $y_4^{(c)} = 0.3047$

9) Apply the milne's predictor corrector method to compute $y(0.4)$ given that $y'' = 1 + y'$

X	0	0.1	0.2	0.3
Y	1	1.1103	1.2427	1.3990
$y' = z$	1	1.2103	1.4427	1.6990

Ans: $y_4^{(p)} = 1.5835$; $z_4^{(p)} = 1.9835$; $y_4^{(c)} = 1.5835$

10) Apply the milne's predictor corrector method to compute $y(0.8)$ given that $y'' = 2yy'$

X	0	0.2	0.4	0.6
Y	0	0.2027	0.4228	0.6841
$y' = z$	1	1.041	1.179	1.468

$Ans : y_4^{(p)} = 1.0237$; $z_4^{(p)} = 2.0304$; $y_4^{(c)} = 1.02823$

11) Apply the milne's predictor corrector method to compute $y(0.4)$ given that $y'' + 3xy' - 6y = 0$

X	0	0.1	0.2	0.3
Y	1	1.03995	1.1380	1.2987
$y' = z$	0.1	0.6995	1.2580	1.8730

$Ans: y_4^{(p)} = 1.5183; z_4^{(p)} = 2.5236; y_4^{(c)} = 1.5138$

12) Apply the milne's predictor corrector method to compute $y(0.4)$ given that $y'' + xy' + y = 0$

X	0	0.1	0.2	0.3
Y	0	0.995	0.9802	0.956
$y' = z$	0	-0.0995	-0.196	-0.2863

$Ans: y_4^{(p)} = 0.9231; z_4^{(p)} = 0.9232; y_4^{(c)} = -0.3692$

****