MODULE 1

FIRST-ORDER FIRST-DEGREE DIFFERENTIAL EQUATIONS

Content:

- Ordinary differential equations: First-order first-degree ordinary differential equations-application to solve simple engineering and scientific problems.
- Numerical solution of first order and first degree ordinary differential equations: Errors and approximations, order of convergence, Modified Euler's method, and Runge - Kutta fourth order method to solve simple engineering and scientific problems.

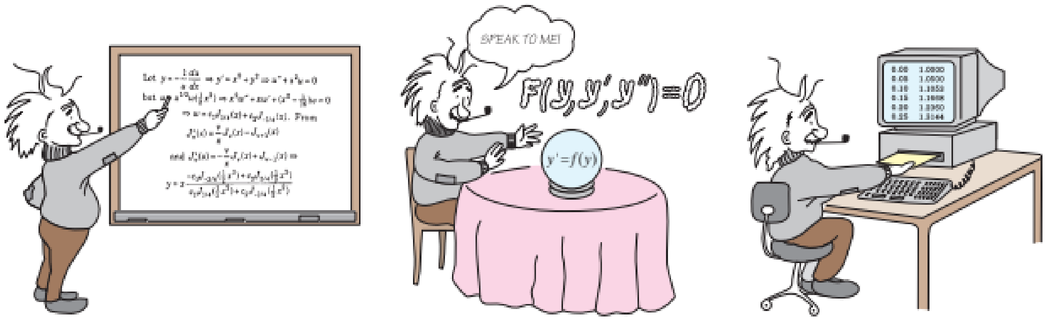
Lab component: **Using MATLAB**

- Solve a first-order first-degree ordinary differential equation analytically by using the inbuilt function and analyze the solution graphically.
- Solve first-order first-degree ordinary differential equations by Runge - Kutta fourth order method.

Learning Objective

- Establish a foundation of modeling dynamical systems to obtain their solution by Analytical and Numerical methods.
- Lay a strong foundation to perform computations of the learned Mathematical concepts using MATLAB.

Imagine you have Rs 1,50,000 and want to invest it at an annual interest rate of 10.5%, compounded continuously. You want to know how much money you will have after a certain number of years (ex. 25 years). How would you calculate this?



(a) analytical **(b)** qualitative **(c)** numerical

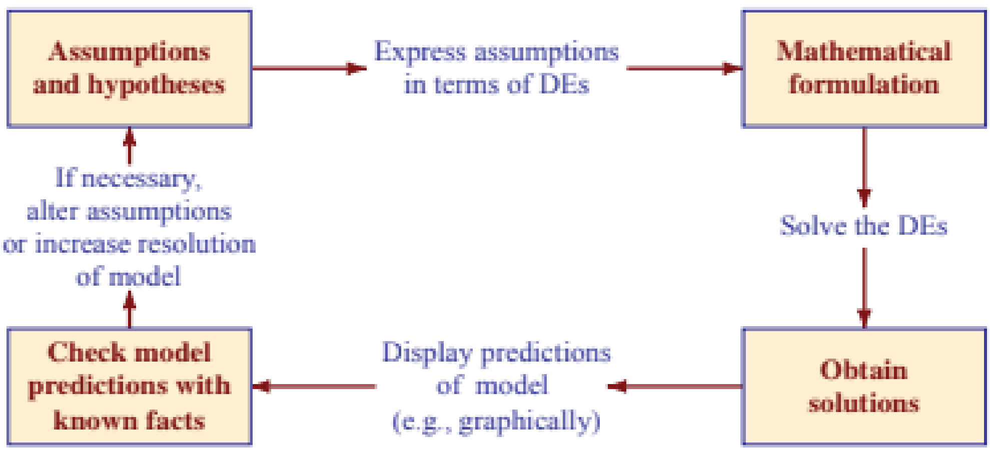
Above Figure Shows Different Approaches to the study of Differential Equations

Mathematical Model

Definition: It is often desirable to describe the behaviour of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system or a phenomenon is called a **mathematical model.**

Construction of a mathematical model of a system starts with *identification of the variables* that are responsible for changing the system.

Figure below shows the Steps in the modelling process with differential equations



- Formulate the real-world problem by translating it into a mathematical expression. Identify and name the independent and dependent variables, and make necessary assumptions to simplify the phenomenon for mathematical analysis. Utilize both your understanding of the physical situation and mathematical skills to derive equations that establish relationships among the variables

- Apply mathematical principles to solve the formulated problem, using our existing knowledge in mathematics to draw mathematical conclusions

- Interpret the mathematical conclusions, extracting insights about the original real-world phenomenon by providing explanations or making predictions based on the derived solutions

- Evaluate our predictions by comparing them against new real data through testing

If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

Examples:

- A constant electromotive force E volt is applied to a circuit containing constant resistance R ohms in series and constant inductance L henries. Can you find how the current changes in the circuit.

- Suppose you are a biologist studying the population dynamics of fish in a lake. Your goal is to model how the fish population changes over time, taking into account factors such as reproduction, predation and environmental conditions.

- Suppose you have a freshly baked pizza that has just come out of the oven. You want to know how long it will take for the pizza to cool down to a safe eating temperature.

The Examples explained above are real world scenarios of

- Electric circuits: R L series circuits
- Population Model: Growth and Decay
- Newton's law of cooling

which are modeled as First-Order and First-Degree Differential Equations and are solved using the following methods

- Variable Separable

- Linear Differential Equation

Ordinary Differential Equations

Definition: An ODE is an equation that contains one or several derivatives of an unknown function y(x) (or sometimes y(t) if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants.

Solution Of Differential Equation

- General Solution
- Particular Solution

General Solution: Solution containing an arbitrary constant c is called General Solution.

Particular Solution: Solution does not contain any arbitrary constants (Specific value of C in General Solution is Chosen) is called Particular Solution

Types of DEs and Solutions of them:

1. Variable Separable method
2. Homogeneous DE
3. Linear DE
4. Bernoulli's DE.
5. Exact DE

1. Variable Separable method

Suppose the given Differential equation is in the form

$$\frac{dy}{dx} = f(x)g(y)$$

By separating the variables, it can be written in the form

$$f(x)dx = \frac{dy}{g(y)}$$

By Integrating the LHS with respect to x and RHS with respect to y, the required solution will be obtained.

Example: Solve $\frac{dy}{dx} = xy$

Solution: Given DE can be written as $\frac{dy}{y} = xdx$

On integration, $\int \frac{dy}{y} = \int x dx$

$\log y = \frac{x^2}{2} + c$, is the required solution.

Example: Solve $\frac{dy}{dx} = x^2 e^{-y}$ **Solution:** $e^y = \frac{x^3}{3} + c$

Example: Solve $\frac{dy}{dx} = (1 + y^2)x$ **Solution:** $tan^{-1}y = \frac{x^2}{2} + c$

Example: Solve $\frac{dy}{dx} = 3y$ **Solution**:$y = ce^x$

2. Linear Differential Equation:

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \qquad \text{.............(1)}$$

Where P and Q are functions of x only or may be constants.

Equation (1) is also known as **Leibnitz's** linear equation.

To solve it, we multiply both sides by$e^{\int P\,dx}$, by getting

$$\frac{dy}{dx} e^{\int P\,dx} + y\left(e^{\int P dx} P\right) = Q e^{\int P\,dx}$$

$$\frac{d}{dx}\left(y e^{\int P\,dx}\right) = Q e^{\int P\,dx}$$

Integrating both sides, we have $ye^{\int P\,dx} = \int Q e^{\int P\,dx}\, dx + c$ which is the required solution

Note :

1. The factor$e^{\int P\,dx}$on multiplying by which the L H S of (1) becomes the differential co-efficient of a single function is called the integrating factor (briefly written as $I.F.$) of (1).

   Thus $I.F = e^{\int P\,dx}$and the solution is $y(I.F.) = \int Q(I.F)\, dx + c$.

2. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as$\frac{dx}{dy} + Px = Q$

where P, Q are functions of y only or constants. The integrating factor in this case is $e^{\int P\,dy}$ and the solution is $x(I.F.) = \int Q(I.F.)\,dy + c$.

Solve: $\frac{dy}{dx} + \frac{y}{x} = x^3$

Solution: The given differential equation is in the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}, Q = x^3$

IF= $e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

The solution is $y(I.F.) = \int Q(I.F.)\,dx + c$

$$yx = \int x^3 x\,dx + c$$

ie $yx = \frac{x^5}{5} + c$

Example: Solve $xdy - (y + 2x^2)dx = 0$ **Solution:** $\frac{y}{x} = 2x + c$

Example: Solve $\frac{dy}{dx} + 2xy = 6e^{x^2}$ **Solution:** $y = \frac{1}{e^{x^2}} \int e^{2x^2} dx$

Example: Solve $\frac{dy}{dx} + 3x^2 y = 6x^2$ **Solution:** $y = 2 + ce^{-x^3}$

Initial value Problem (IVP):

An IVP is a concept in the context of DE, which involves solving a DE with the additional requirement of specifying the values of the unknown function and its derivatives at a particular point, usually the starting point of the interval.

1. Solve $\frac{dy}{dx} = -\frac{x}{y}, y(4) = 3$ **Solution**: $x^2 + y^2 = 25$
2. Solve $\frac{dy}{dx} - 3y = 6, y(0) = 0$ **Solution**: $y = -2 + ce^{3x}$
3. Solve $\frac{dy}{dx} = xy^{\frac{1}{2}}, y(0) = 0$ **Solution**: $y = \frac{x^4}{16}$

1.1 Electrical Circuits: L R Series Circuits

Modelling electric circuits using differential equations is a common approach in electrical engineering and physics.

Modeling a resistor (R) and inductor (L) in series in an electrical circuit involves understanding the relationship between voltage (V), current (I), resistance (R), and inductance (L) in the circuit. The key equations are derived from Ohm's Law and the behavior of inductors.

1. **Ohm's Law:** $V = I \cdot R$

This equation relates voltage (V), current (I), and resistance (R) in any electrical circuit.

2. **Inductor Voltage:** The voltage across an inductor is given by $V_L = L\frac{dI}{dt}$

Where V_L is the voltage across the inductor, L is the inductance, $\frac{dI}{dt}$ is the change in current with respect to time.

For an RL series circuit, you need to consider both the resistor and inductor. The total voltage (V_{total}) across the series RL circuit is the sum of the voltage across the resistor (V_R) and the voltage across the inductor (V_L)

$$V_{total} = V_R + V_L$$

Given Ohm's Law and the inductor voltage equation,

$$V_{total} = IR + L\frac{dI}{dt}$$

This is a first-order linear differential equation that describes the behavior of an RL circuit. Solving this equation can give you the relationship between current, voltage, and time in the RL circuit.

Note: If Q be the electrical charge on a condenser of capacity C and i be the current, then

(a) $i = \frac{dQ}{dt}$ or $Q = \int i\, dt$

(b) The potential drop across the resistance R is Ri.

(c) The potential drop across the inductance L is $L\frac{di}{dt}$.

(d) The potential drop across the capacitance C is $\frac{Q}{C}$.

Also, by Kirchhoff's Law, the total potential drop (voltage drop) in the circuit is equal to applied voltage (E.M.F.).

1.2 Population Model

Population models describe the changes in the size of a population over time. Using differential equations to model population dynamics is a common approach in Population Model. The two main types of population models are the exponential growth model and the logistic growth model.

1. Exponential Growth Model:

The exponential growth model assumes that the rate of growth of a population is proportional to its current size. This leads to a simple first-order linear differential equation.

Let $P(t)$ be the population at time t, and k be the growth rate. Then

$$\frac{dP}{dt} = kP$$

Here:

- $\frac{dP}{dt}$ is the rate of change of population with respect to time.
- k is the growth rate constant.
- $P(t)$ is the population at time t.

This differential equation has a solution of the form: $P(t) = P_0 e^{kt}$ where P_0 is the initial population at $t = 0$.

2. Logistic Growth Model:

The logistic growth model introduces a carrying capacity (K), representing the maximum population size that the environment can support. This model incorporates the idea that as the population approaches the carrying capacity, the growth rate decreases.

The logistic growth equation is a nonlinear differential equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

- r is the intrinsic growth rate.
- $P(t)$ is the population at time t.
- K is the carrying capacity.

This differential equation has a solution of the form:

$P(t) = \frac{K}{1+\frac{K-P_0}{P_0}e^{-rt}}$ where P_0 is the initial population at $t = 0$.

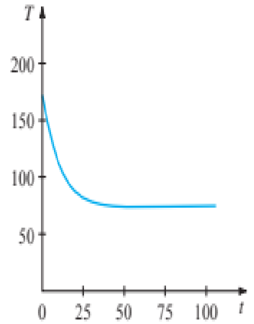
1.3 Newton's law of cooling

Newton's Law of Cooling describes the rate of change of the temperature of an object as it cools or heats in a surrounding environment. It is often used to model heat transfer processes. The law states that '**the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature'.**

Let T(t) represent the temperature of an object at time t, and let T_{env} be the ambient temperature. Then the differential equation for Newton's Law of Cooling can be expressed as follows:

$$\frac{dT}{dt} = -k\,(T(t) - T_{env})$$

- $\frac{dT}{dt}$ is the rate of change of temperature with respect to time.
- T(t) is the temperature of the object at time t.
- T_{env} is the ambient temperature.
- k is the cooling or heating constant, which depends on the specific properties of the object and the surrounding medium.



Above Graph Represents Cooling Curve

Problems:

1. Rabbit Population Dynamics

Suppose you have a population of rabbits on an isolated island. The population of Rabbit can be described as follows.

Rabbits reproduce at a constant rate of 0.2 per year per rabbit.

Rabbits experience a death rate that is proportional to their current population, with a constant proportionality of 0.1 per year.

At the start, you have 100 adult rabbits on the island. Find the rabbit population dynamics over time.

Solution:

Let R(t) represent the population of adult rabbits at time t (in years). The rate of change of the rabbit population can be described by the following differential equation:

$\frac{dR}{dt}$ = Birth Rate R(t) - Death Rate R(t)

$\frac{dR}{dt} = 0.2R(t) - 0.1R(t)$

Now, you can solve this differential equation to model how the rabbit population changes over time.

To solve this equation, you can use methods like **separation of variables** and the solution will provide you with the population of adult rabbits as a function of time. **Solution**: $P = ce^{0.1t}$

2. Bacterial Growth

A culture initially has P_0 number of bacteria. At t = 1 h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria P(t) present at time t, determine the time necessary for the number of bacteria to triple.

Solution: We first solve the differential equation in (1), with the symbol y replaced by P. With $t_0 = 0$ the initial condition is $P(0)=P_0$.

We then use the empirical observation that $P(1)=\frac{3}{2}P_0$ to determine the constant of proportionality k.

Notice that the differential equation $\frac{dP}{dt} = kP$ is both separable and linear.

When it is put in the standard form of a linear first-order DE

$$\frac{dp}{dt} = kP = 0$$

we can see by inspection that the integrating factor is e^{-kt},Multiplying both sides of the equation by this term and integrating gives, in turn,

$$\frac{d}{dt}[e^{-kt}P] = 0 \text{ and } e^{-kt}P = c$$

Therefore $P(t) = ce^{kt}$. At $t = 0$ it follows that $P_0 = ce^0 = c$, so $P(t) = P_0e^{kt}$. At $t = 1$ we have $\frac{3}{2}P_0 = P_0e^k$ or $e^k = \frac{3}{2}$. From the last equation we get, $k = ln\frac{3}{2} = 0.4055$, so $P(t) = p_0e^{0.4055t}$

To find the time at which the number of bacteria has tripled, we solve $3P_0 = p_0e^{0.4055t}$for t. It follows that $0.4055t = ln\,3$, or $t = \frac{ln\,3}{0.4055} = 2.71\,h$= 2h 42.36min

3. A constant electromotive force E volt is applied to a circuit containing constant resistance R ohms in series and constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $\frac{L\log 2}{R}$ seconds.

Solution: Let i be the current in the circuit at any time t.

By Kirchhoff's law, we have, $L\frac{di}{dt} + Ri = E$ or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$……..(1)

which is a Leibnitz's linear equation, $I\,F = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

∴ the solution of equation (1) is $i(I\,.\,F) = \int \frac{E}{L}(I.F)dt + c$

$$i.e\ e^{\frac{Rt}{L}} = \int \frac{E}{L}.e^{\frac{Rt}{L}}\,dt + c = \frac{E}{L} + \frac{L}{R}e^{\frac{Rt}{L}} + c$$

$$\text{Or } i = \frac{E}{R} + ce^{\frac{Rt}{L}}$$

Initially, when $t = 0$, $i = 0$ so that $c = -\frac{E}{R}$

Thus (2) becomes, $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ ……..(3)

This equation gives the current in the circuit at any time t.

Clearly, i increases with t and attains the maximum value $\frac{E}{R}$.

Let the current in the circuit be half its theoretical maximum after a time T seconds.

Then

$$\frac{1}{2}.\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \text{ or } e^{\frac{Rt}{L}} = \frac{1}{2}$$

$$\text{Or} -\frac{RT}{L} = \log\frac{1}{2} = -\log 2$$

$$T = (L\log 2)/R$$

4. A breeder reactor converts relatively stable uranium-238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining. **Solution:** $t = \frac{\ln 2}{0.00002867} \approx 24{,}180yr.$
5. The equations of electromotive force in terms of current i for electrical circuit having resistance R and a condenser of capacity C, in series, is $E = Ri + \int \frac{i}{C}dt$. Find the current i at any time t , when $E = E_0 \sin \omega t$.

Solution: The given equation can be written as $R_i + \int \frac{i}{C}dt = E_0 \sin \omega t$.

Differentiating both sides w.r.t. t, we have $R\frac{di}{dt} + \frac{i}{C} = \omega\ E_0 \sin\omega t$

Or $\frac{di}{dt} + \frac{i}{RC} = \frac{\omega E_0}{R}\cos\omega t$ .... (1)

$I\,F = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$

∴ the solution of equation (1) is

$$e^{\frac{t}{RC}} = \int \frac{\omega E_0}{L}\cos\omega t\ .\, e^{\frac{t}{RC}}\,dt = \frac{\omega E_0}{R} + \int e^{\frac{t}{RC}}\cos\omega t\,dt$$

$$= \frac{\omega E_0}{R}.\frac{e^{\frac{t}{RC}}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\cos\left(\omega t - \tan^{-1}\frac{\omega}{\frac{1}{RC}}\right) + k$$

$$= \frac{\omega C E_0}{\sqrt{1+R^2C^2\omega^2}}.\,e^{\frac{t}{RC}}\cos(\omega t - \ \Phi) + k \quad \text{where } \tan\Phi = RC_\omega \text{Or}$$

$$i = \frac{\omega C E_0}{\sqrt{1+R^2C^2\omega^2}}\cos(\omega t - \ \Phi) + ke^{\frac{t}{RC}},$$

This gives the current at any time$t$.

6. The current i and the charge q in a series circuit containing an inductance L, capacitance C, e.m.f E satisfy the d.e. $L\frac{di}{dt} + Ri = E$. Express q and i in terms of t, given that L, C, E are constants and the value of I,q are both zero initially.

Solution: q(t)=EC[1-cos$\sqrt{1/LC}$ t], Also i(t)=$q'(t)$ = E$\sqrt{C/L}$ sin$\sqrt{1/LC}$ t.

7. Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 grams of uranium are present at timesT_1 and T_2respectively, show that the half-life of uranium is$\frac{(T_2 - T_1)\log 2}{\log(\frac{M_1}{M_2})}$

Solution: Let M grams of uranium be present at any time. Then the equation of disintegration of uranium is

$$\frac{dM}{dt} = -kM, \text{where k is a constant}.$$

Or $\frac{dM}{M} = -k\,dt$

Integrating, $\log M = \ -kt + c$ ....(1)

Initially, when $t = 0, M = M_0$(say)

:. From (1), $c = \log M_0$

Substituting the value of c in (1), we have $\log M = \log M_0 - kt$

Or $\quad kt = \log M_0 - \log M$ ……..(2)

Now, when $t = T_1$, $M = M_1$ and when $t = T_2$, $M = M_2$

from (2) , we have

$kT_1 = \log M_0 - \log M_1$ ………..(3)

$kT_2 = \log M_0 - \log M_2$ ………..(4)

Subtracting (3) from (4) , we got

$$k(T_2 - T_{1)} = \log M_1 - \log M_2 \quad \text{or} \quad k = \frac{\log\frac{M_1}{M_2}}{T_2 - T_1}$$

Let T be the half life of uranium i.e., when $t = T, M = \frac{1}{2} M_0$

From (2), we get $kT = \log M_0 - \log\frac{M_1}{2} = \log 2$

$$T = \frac{\log 2}{k} = \frac{(T_2 - T_1)\log 2}{k\log\left(\frac{M_1}{M_2}\right)}$$

8. Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10PM and turned on again at 6AM. On a certain day the temperature inside the building at 2AM was found to be 65°F. The outside temperature was 50°F at 10PM and had dropped to 40°F by 6AM. What was the temperature inside the building when the heat was turned on at 6AM?

Solution: Setting up a model: let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's law).

$$\frac{dT}{dt} = k(T - T_A)$$

General solution: For constant T_A=45, the ODE is separable, Separation, integration, taking exponents gives the general solution.

$$-\frac{dT}{T-45} = kdt$$

$$\ln|T - 45| = kt + c^*$$

$$\mathrm{T(t) = 45 + ce^{kt}}$$

Particular Solution: We choose at 10AM, $t = 0$. Then $T(0) = 70$

$$T(0) = 45 + ce^0$$

$$\mathrm{c = 25}$$

$$\mathrm{T_P(t) = 45 + 25e^{kt}}$$

Determination of k: We use $\mathrm{T(4) = 65}$, where $\mathrm{t = 4}$ at 2AM.

$$\mathrm{T_P(t) = 45 + 25e^{4k} = 65}$$

$$\mathrm{e^{4k} = 0.8}$$

$$\mathrm{k = -0.056}$$

$$T_P(t) = 45 + 25e^{-0.056t}$$

9. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P0 has doubled in 5 years, how long will it take to triple? To quadruple? **Solution:** 7.9 yr; 10 yr.
10. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay? **Solution:** 11h
11. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t.3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria? **Solution**: P_0 =200.67876
12. A thermometer, reading 5℃, is brought into a room whose temperature is 22℃. One minutes later the thermometer reading is12℃. How long does it take until the reading is practically 22℃, say21.9℃? **Solution:** $t = 9.68min$
13. If the temperature of the cake is 300°F when it leaves the oven and is 200°F ten minutes later, when will it be practically equal to the room temperature of 60°F, say when will it be 61°F? **Answer**: t=102min
14. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° in 1 second? How long will it take the bar to reach 98° C? **Solution**: 39seconds

15. An RL circuit has an emf of 5V, resistance of 50Ω, an inductance of 1H, and no initial current. Find the current in the circuit at any time t. What is the limiting value of the current. Solution: $i = 0.1(1 - e^{-50t})A$

16. Solve the resulting ODE for the current I(t) A, where t is time. Assume that the circuit contains an EMF E(t)=48 V, a resistor 11 ohms, an inductor 0.1 H, and that the current is initially zero. When does the current reaches its steady state. **Solution**: $I = \frac{48}{11}(1 - e^{-110t})$

17. Suppose $5000 is deposited in to an account which earns continuously compounded interest. Under these conditions, the balance in the account grows at a rate proportional to the current balance. Suppose that after 4 years the account is worth $7000.

    a. How much is the account worth after 5years. **Solution**=$7614.30

    b. How many years does it take for the balance to double? **Solution**=8.24yrs

18. After 10 minutes in a room, a cup of tea is cooled to 40^0 C from 100^0 C. The room temperature is 25^0 C. How longer will it take to cool to 35^0 C? **Solution**=2.124minutes.

19. A 12 volt battery is connected to a series circuit in which the inductance is 0.5H and the resistance is 10ohms. Determine the current i if the initial current is zero. **Solution:**$\boldsymbol{i(t) = \frac{E_0}{R} + ce^{-(R/L)\,t}}$

Note: When analytical methods fail due to the inherent complexities of the equations or the specific conditions of the problem Numerical methods, in particular, are used for solving such equations because they can provide accurate solutions.

Numerical Solution of First Order ODE:

Numerical methods are used whenever mathematical problems cannot be solved analytically or when the analytical solutions are too complex or computationally expensive to obtain. They are fundamental tool in modern science and engineering for solving practical problems across a wide range of disciplines.

Used extensively in Financial Modelling, Statistical Analysis, Image and Signal Processing task, Machine Learning and Data Analysis.

Euler's & modified Euler's method

An ODE of first order is of the form $F(x, y, y_1) = 0$. An Initial Value Problem of this equation is of the form $y_1 = f(x, y), y(x_0) = y_0$.

The initial approximation for y_1 at $x_1 = x_0 + h$ is found using **Euler's** formula, which is given by $y_1^{(0)} = y_0 + hf(x_0, y_0)$

The other approximations are found using **Modified Euler's** Formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right)$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right)$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right)$$ And so on.

1. Consider the initial-value problem $y' = 0.1y^{1/2} + 0.4x^2$, $y(2) = 4$. Use Euler's method to obtain an approximation of $y(2.5)$ using first $h = 0.1$ and then $h = 0.05$.

SOLUTION: On solving we get

When $h = 0.1$

x_n	y_n
2.00	4.0000
2.10	4.1800
2.20	4.3768
2.30	4.5914
2.40	4.8244
2.50	5.0768

When $h = 0.05$

x_n	y_n
2.00	4.0000
2.05	4.0900
2.10	4.1842
2.15	4.2826
2.20	4.3854
2.25	4.4927
2.30	4.6045
2.35	407210
2.40	4.8423
2.45	4.9686
2.50	5.0768

We see in Tables that it takes five steps with $h = 0.1$ and 10 steps with $h = 0.05$, respectively, to get to $x = 2.5$. Intuitively, we would expect that $y_{10} = 5.0997$ corresponding to $h = 0.05$ is the better approximation of $y(2.5)$ than the value $y_5 = 5.0768$ corresponding to $h = 0.1$.

So The Value of "h" plays a crucial role in determining the accuracy and efficiency of these numerical methods.

When the step size "h" is decreased the given interval is dividing into smaller subintervals. This results in a more detailed approximation of the function. ***Smaller "h" values generally lead accurate numerical results***

Larger step sizes result in coarser approximations. The numerical method may not capture small-scale variations or features in the function. ***Larger "h" values reduces accuracy***

Errors and Approximation

Error Analysis: The numerical error is the difference between the exact solution and the approximation solution type of error.

1. **Inherent Error:** Inherent error exist in the problem either due to approximate given data, limitation of the computing aids such as mathematical tales, desk calculator etc, Such type of error can be reduced by taking better data and using precision computing aids .
2. **Round off error:** These error are due to rounding off the number

Ex: 1.24372≈ 1.244 Error: 1.244 - 1.24372 =0.00028

3. **Truncation Error:** The error arises due to use of truncating the infinite series to some approximate terms.
   Ex: Exponential series, Sinx, etc.
4. **Absolute Error:** Absolute Error is the positive difference between the actual value and the approximate value.
   Absolute Error = $|actual\ value - approximation|$
5. **Relative Error:** $\frac{absolute\ error}{actual\ value}$
6. **Percentage Error:** $\frac{absolute\ error}{actual\ value} * 100$

Order of Convergence

The order of convergence is a concept used in numerical analysis to describe how quickly an iterative numerical method approaches the desired solution. It is often expressed as a rate at which the error decreases with each iteration. There are several different orders of convergence, including linear, quadratic, cubic, and so on. Higher orders of convergence generally indicate faster convergence, which means the numerical method approaches the true solution more quickly with each iteration. However, achieving higher orders of convergence may require additional computations or more complex algorithms.

Problems:

1. Consider the initial-value problem $y = 0.2xy,\ y(1) = 1$. Use Euler's method to Obtain an approximation of $y(1.5)$ using first $h = 0.1$ and then $h = 0.05$.

Solution: The true or actual values were calculated

x_n	y_n	Actual Value	Abs. error	%Rel. error
1.00	1.0000	1.0000	1.0000	0.00
1.10	1.0200	1.0212	1.0012	0.12
1.20	1.0424	1.0450	1.0025	0.24
1.30	1.0675	1.0714	1.0040	0.37
1.40	1.0952	1.1008	1.0055	0.50
1.50	1.1259	1.1331	1.0073	0.64

2. Apply modified Euler's Method to the initial value problem $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$, find $y(0.1)$, by taking h=0.05

Solution: $f(x, y) = x^2 + y^2, h = 0.05$

Stage I: $x_0 = 0, y_0 = 1 f(x_0, y_0) = 1, x_1 = 0.05$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1.05$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right) = 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right) = 1.0513$$

$y(0.05) = 1.0513$

Stage II: $x_0 = 0.05, y_0 = 1.0513 f(x_0, y_0) = 1.0538, x_1 = 0.1$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1.104$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right) = 1.1055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right) = 1.1055$$

$y(0.1) = 1.1055$

3. Apply modified Euler's Method to the initial value problem $\frac{dy}{dx} = x^2 + y^2, y(0) = 0,$ find $y(0.1)$ **Solution:** y(0.1) = 0.0005

4. $(i) y'$ =y, y(0) = 1; y(1.0) ,
(ii) y' =2xy, y(1) = 1; y(1.5)

Use Euler's method to obtain a four-decimal approximation of the indicated value. First use h = 0.1 and then use h = 0.05**.**Find Actual value, Absolute error and % error.

5. Given $\frac{dy}{dx} - \sqrt{xy} = 2$, y(1)=1, find y(1.25) using Modified Euler's method, take h=0.25. **Solution**: y(1.25)=1.8132

6. Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, y(1)=2, find the approximate value of y at x=1.2 by taking step size h=0.2 by applying Modified Euler's method. **Solution**: y(1.2)=2.6182

7. Using Euler's and Modified Euler's method compute y(2.5) with h=0.25 from $\frac{dy}{dx} = \frac{(x+y)}{x}, y(2) = 2.$ **Solution**: y(2.25)=2.5147, y(2.5)=3.0572

RUNGE-KUTTA METHODS:

Runge-Kutta methods are more accurate methods of great practical importance. They do not require the computation of higher order derivatives as in Taylor's series method, rather they require evaluation of $f(x, y)$. They agree with Taylor's series up to the terms of h^r, where r is different for different methods and is known as the order of that Range-Kutta method.

Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of first, second and third order respectively. **The fourth order Runge-Kutta method** is most commonly used in practice and is often referred to as **'the Runge-Kutta method'** only without any reference to the order.

To solve the differential equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

By Runge –kutta method,To compute y_1

$k_1 = hf(x_0, y_0);$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right);$$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + 2k_4),$

$y_1 = y(x_0 + h) = y_0 + k.$

To compute y_2 ,

$k_1 = hf(x_1, y_1),$

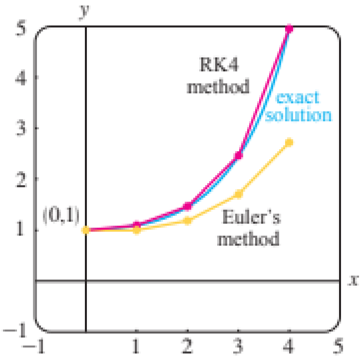
$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right),$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4);$

$y_2 = y_1 + k$, and so on.



Comparison Between Euler's Method And RK-Method

Problems:

1. Use Runge-Kutta method to find y when x = 1.1, 1.2 in steps of 0.1 given that $\frac{dy}{dx} = x^2 + y^2$ and $y(1) = 1.5$

Sol. Here $f(x, y) = x^2 + y^2$, $h = 0.1$

For $y(1.1)$, we have $x_0 = 1$, $y_0 = 1.5$

$k_1 = h(x_0, y_0) = 0.1\, f(1, 1.5) = 0.1(1^2 + (1.5)^2) = 0.325$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(1 + 0.05, 1.5 + 0.1625) = 0.1(1.05^2 + 1.66255^2) = 0.3866$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(1 + 0.05, 1.5 + 0.1933) = 0.1(1.05^2 + 1.6933^2) = 0.3969$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(1 + 0.1, 1.5 + 0.3969 = 0.1(1.1^2 + 1.8969^2) = 0.4808$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(1.1) = 1.5 + \frac{1}{6}(0.325 + 0.7732 + 0.7938 + 0.4808) = 1.8954$

To find $y(1.2)$, we have $y(1.1) = 1.8954$

$x_1 = 1.1, y_1 = 1.8954$

$k_1 = hf(x_1, y_1) = 0.1f(1.1, 1.8954) = 0.1(1.1^2 + 1.8954^2) = 0.4802$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(1.1 + 0.05, 1.8954 + 0.2401) = 0.1(1.15^2 + 2.1355^2)$
$= 0.5882$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(1.1 + 0.05, 1.8954 + 0.2941) = 0.1(1.2^2 + 2.1895^2) = 0.6116$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(1.1 + 0.1, 1.8954 + 0.6116) = 0.1(1.2^2 + 2.5070^2)$
$= 0.7725$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(1.2) = 1.8954 + \frac{1}{6}(0.4802 + 1.1764 + 1.2232 + 0.7725) = 2.5041$

1. Given $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, using Runge-Kutta method of Fourth order.

Solution: Here $k_1 = hf(x_0, y_0) = -0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f[0.05, 0.95] = (0.1)[(0.05)^2 - 0.95] = -0.09475$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f[0.05, 0.952625] = (0.1)[(0.05)^2 - 0.952625]$
$= -0.0950125$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf[0.1, 0.9049875] = (0.1)[(0.1)^2 - 0.9049875]$
$= -0.0894987$

Now

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125) - 0.0894987$$
$$= -0.0948372$$

$$y_1 = y(0.1) = y_0 + k = 1 - 0.0948372 = 0.9051627$$

2. Solve $(y^2 - x^2)dx = (y^2 + x^2)dy$ for x=0.2 given that y=1 at x=0 initially, by applying R-K method of order four.

   Solution: k_1=0.2, k_2=0.1967, k_3=0.1967, k_4=0.1891, y(0.2)=1.19598

3. Use the RK4 method with h 0.1 to obtain an approximation to y(1.5) for the solution of y^1 = 2xy, y(1) =1. **Solution:** y(1.5)=1.2337

4. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, y(0)=1, find y(0.1) using Runge-Kutta method of Fourth order(considering up to 3 decimal places). **Solution**: y(0.1)=1.067

5. Use Runge- Kutta method to find y when x= 1.1 in steps of 0.1 given that $\frac{dy}{dx} = x^2 + y^2, y(1) = 1$ **Solution**: y(1.1) =1.2342

6. Use Runge- Kutta method to find y when x= 0.1 given that $\frac{dy}{dx} = 3e^x + 2y$, with y(0)=0. **Solution**: y(0.1)=0.3487

7. Given $\frac{dy}{dx} = x + y^2, y(0) = 1$, Estimate the value of y(0.1) Using RK method of order four by considering up to third decimal place. **Solution**: y(0.1)=1.117

8. Consider an LR circuit with L=0.2H, R=10ohms, and an applied voltage of V(t)=12V. If the initial current is i(0)=0 use RK method of order 4 to simulate the current through the circuit for 5 seconds. **Solution**: i(5)=-192224700

9. Suppose an object is initially at 100^0 c in a room at 25^0 c, with a cooling co-efficient of 0.1 per minute. Use R-K method of order four to approximate the temperature of the object over 30minutes .

10. Suppose you are studying the population growth of species of birds in a forest. You have collected data indicating that the intrinsic growth rate (r) is 0.2per year, and the carrying capacity(K) of this species is 5000 birds. Model the population growth of these birds over 10 year period using logistic growth model (consider the initial population to be 1000) and solve using RK method.

11. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is determined that the rate at which the virus spreads is proportional not only to the number T(t) of students infected but also to the number of students not

infected. Determine the number of infected students after 6days given that the number of infected students after 4days is 50. **Solution**=276Students

Module Outcomes:

- At the end of the course, the student will be able to:
- Illustrate the knowledge of fundamental concepts of first order- first degree ordinary differential equations.
- Apply suitable techniques to solve given engineering and scientific problems related to first order- first degree ordinary differential equations based on the acquired knowledge.
- Analyze mathematical solutions of engineering and scientific problems related to first-order first-degree ordinary differential equations and predict their behavior in real-world scenario.
- Use MATLAB to perform mathematical computation related to first-order first-degree ordinary differential equations.