MODULE-3
MATRIX ALGEBRA

Content:

Introduction to Linear Algebra related to Engineering applications.

- Solution of system of linear equations.
- Elementary row transformation of a matrix
- Rank of a matrix
  - Gauss-elimination method
  - approximate solution by Gauss-Seidel method.

Solution of system of Ordinary Differential equations by Matrix method.

Lab component: **Using MATLAB**

- Solve system of linear equations by inbuilt function and Gauss-Seidel iterative method
- Solve system of linear ordinary differential equations by inbuilt function and matrix method.

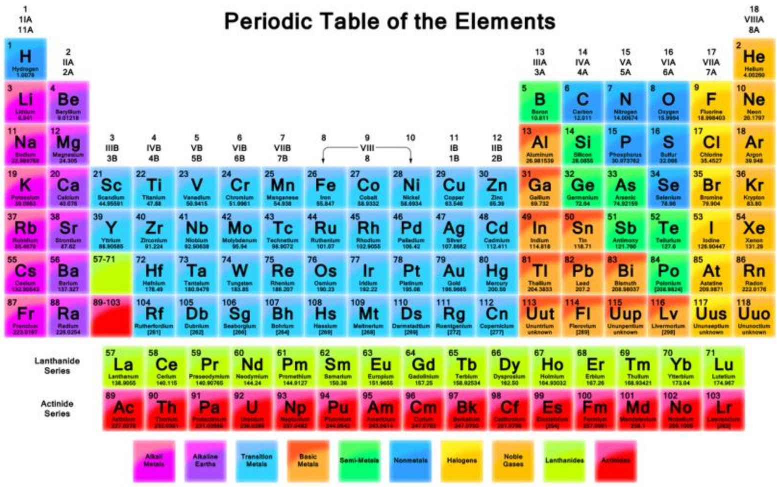
Learning Objective

- Impart knowledge of various matrix methods and techniques for solving system of linear equations & ordinary differential equations.
- Lay a strong foundation to perform computations of the learned mathematical concepts using MATLAB.

Observe the following images and tell me what you can infer from this

Seating arrangement in auditorium.

Pixel Image



Marks list of Students

	A	B	C	D
1	Student	Exam 1	Exam 2	Exam 3
2	Jim	70	72	80
3	Sue	78	85	99
4	Andy	89	92	88
5	Bo	90	88	86
6	Aki	81	67	78
7				

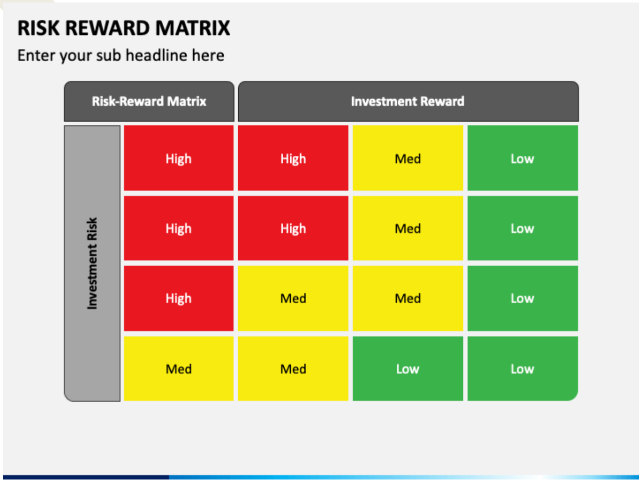
INTRODUCTION:

A matrix is a fundamental mathematical concept used to organize and manipulate data in various fields, including mathematics, physics, computer science, and engineering. It consists of rows and columns of numbers, symbols, or variables arranged in a rectangular grid. Matrices are employed for a wide range of applications, such as solving systems of linear equations, representing transformations in computer graphics, and conducting statistical analyses. They play a crucial role in linear algebra, providing a powerful tool for representing and solving complex mathematical problems through operations like addition, multiplication, and inversion. Matrices are integral to many aspects of modern science and technology, making them an essential concept for anyone working with data or mathematical modelling.

BASIC MODELS AND DEFINITIONS:

In the world of finance, matrices come into play when managing investment portfolios, optimizing asset allocation, and assessing risk. They enable analysts to evaluate the relationships between different financial assets and make informed decisions to maximize returns and minimize potential losses.

In the case of budgeting a company may have multiple sources of income and expenses. They need to balance their budget by solving equations to ensure that income exceeds expenses. In Investment Portfolios, an investor may have investments in different assets with varying returns. They need to calculate how different allocations affect their overall return. This can be analyzed with matrix known as Decision Matrix or Risk-Reward Matrix.

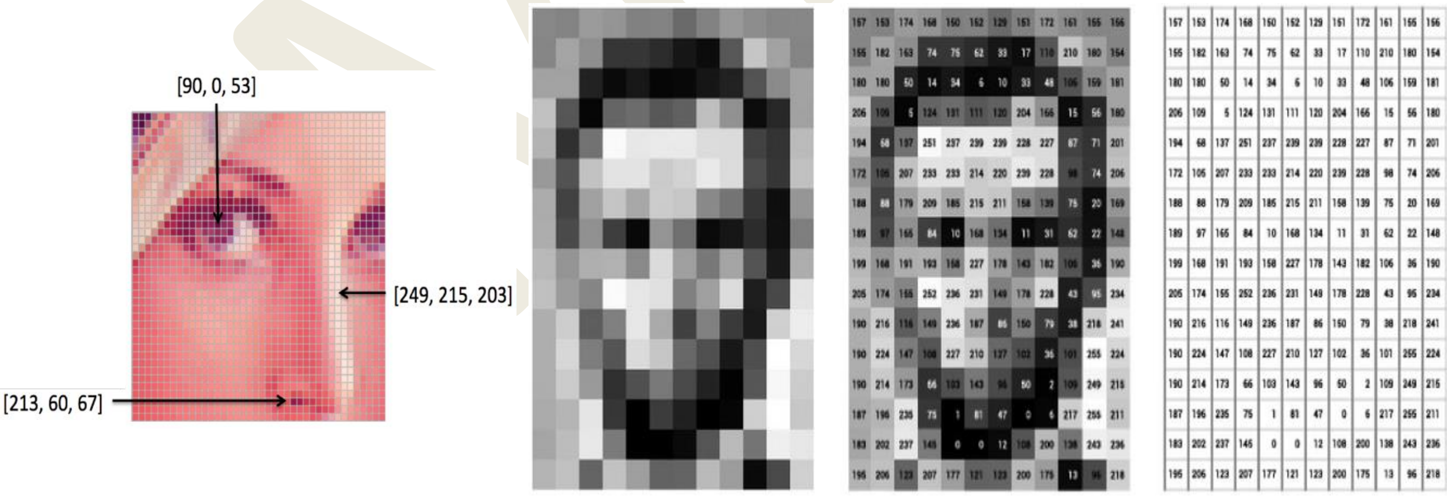


A decision matrix, also known as a decision-making matrix or criteria matrix, is a systematic tool used to evaluate and compare multiple options or alternatives based on a set of criteria or factors. It helps individuals or teams make informed decisions by providing a structured framework for assessing and ranking different choices.

In the case of image processing matrices are used for a number of purposes, including storing and manipulating images, applying image filters, and performing image transformations. They are also used in matrix algebra, which is a branch of mathematics that is useful for simplifying complex image processing operations.

In image processing, a matrix is a two-dimensional array of numbers that represents the pixels in an image. Each element of the matrix corresponds to a single pixel in the image, and the value of the element represents the intensity or color of the pixel.

For example, an image with M rows and N columns can be represented as a matrix with M rows and N columns, where each element of the matrix represents the intensity or color value of a single pixel in the image. This allows the image to be stored and manipulated in a computer. Image filters can be applied by multiplying the pixel values of the image matrix by a filter matrix, and image transformations can be performed by multiplying the pixel values of the image matrix by a transformation matrix.



DEFINITION: A matrix is a rectangular arrangement of numbers in rows and columns represented by

$$\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}.$$

If a matrix has m rows and n columns, then it is said to be of order $m \times n$ (read as "m by n" matrix). The elements a_{ij} of a matrix are identified by double subscript notation ij, where i denotes the row and j denotes the column.

SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS:

System of simultaneous linear equations can be found in various real-life scenarios where multiple variables are related to each other in a linear fashion. Here are some general real-life examples:

1. Suppose Kumari had rupees 12100 to invest. She decided to invest her money in bonds and mutual funds. She invested a portion of the money in bonds paying 8% interest per year and the remainder in a mutual fund paying 9% per year. After one year the total income she had earned from the investments was rupees 1043. How much had she invested at each rate.?

Mathematically, we can solve the above problem by considering it as a system of linear equations. The two quantities in this problem are the amount she had invested in bonds and the amount she had invested in mutual funds. Let the amount invested in bonds be represented by x and the amount invested in mutual funds be represented by y. Then we can express the problem in the form of equations as follows:
The total amount of money she had to invest was

$$x + y = 12100.$$

The amount of money she earned from the investments was

$$\frac{8}{100}x + \frac{9}{100}y = 1043.$$

By solving the above system of equations, we can obtain the money she invested in bonds and mutual funds.

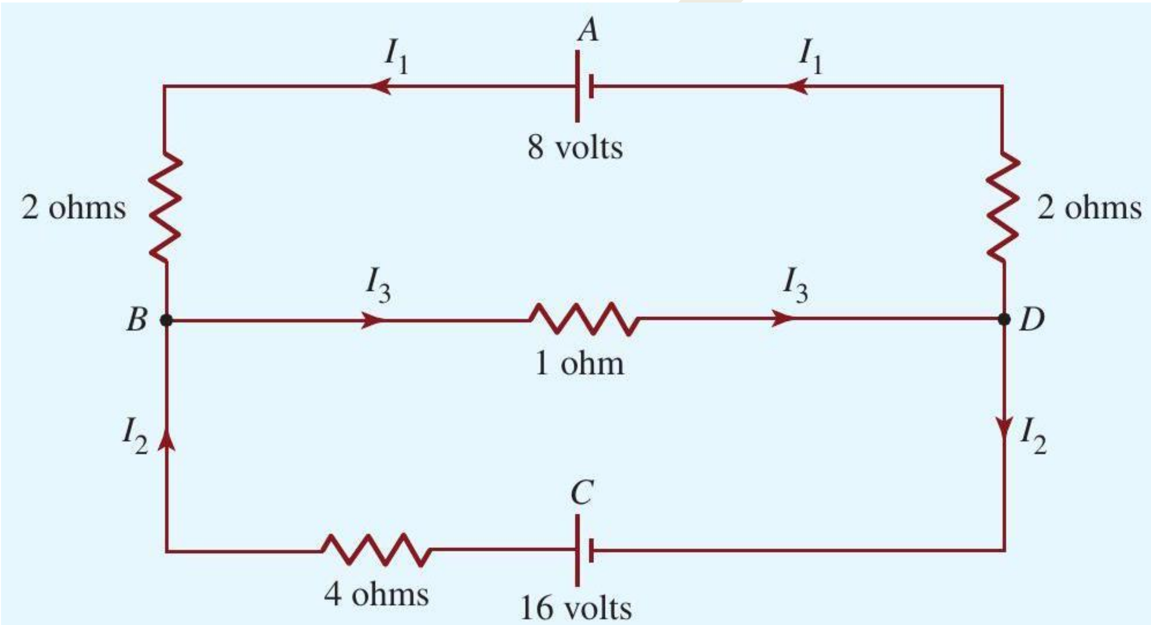
2. Electrical networks are a specialized type of network providing information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source forces a current to flow through the network, where it encounters various resistors, each which requires a certain amount of force to be applied in order for the current to flow through. Systems of linear equations are used to determine the currents through various branches of electrical networks. We know that Ohm's Law is given as the voltage drop across a resistor is given by V = IR and Kirchhoff's Law is given by

• **Junction**: All the current flowing into a junction must flow out of it.

• **Path**: The sum of the IR terms (I denotes current, R resistance) in any direction around a closed path is equal to the total voltage in the path in that direction.

We wish to determine the currents I_1, I_2 *and* I_3 in the below circuit. Applying Ohm's and Kirchhoff's Law, we can construct a system of linear equations.

Example: Consider the following electrical network and determine the currents through each branch of this network.



The batteries are 8 volts and 16 volts. The following convention is used in electrical engineering to indicate the terminal of the battery out of which the current flows: The resistances are one 1-ohm, one 4-ohm, and two 2-ohm. The current entering each battery will be the same as that leaving it.

Let the currents in the various branches of the above circuit be I_1, I_2, and I_3. Kirchhoff's laws refer to junctions and closed paths. There are two junctions in this circuit, namely the points B and D. There are three closed paths, namely $ABDA, CBDC$, and $ABCDA$. Apply the laws to the junctions and paths.

$$\text{Junction } B, \quad I_1 + I_2 = I_3$$
$$\text{Junction } D, \quad I_3 = I_1 + I_2$$

These two equations result in a single linear equation

$$I_1 + I_2 - I_3 = 0.$$

Paths:

$$\text{Path } ABDA, \quad 2I_1 + 1I_3 + 2I_1 = 8$$
$$\text{Path } CBDC, \quad 4I_2 + 1I_3 = 16$$

It is not necessary to look further at path $ABCDA$. We now have a system of three linear equations in three unknowns, I_1, I_2, and I_3. Path $ABCDA$ in fact leads to an equation that is a combination of the last two equations.

The problem thus reduces to solving the following system of three linear equations in three variables.

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 4I_1 + I_3 &= 8 \\ 4I_2 + I_3 &= 16 \end{aligned}$$

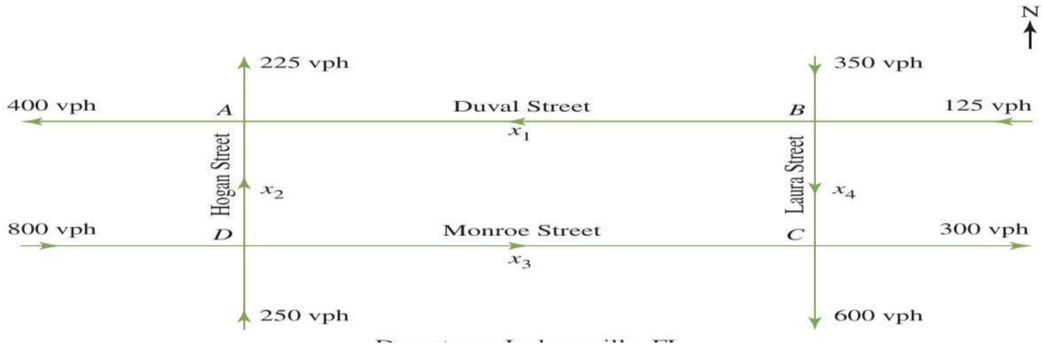
3. The concepts and tools of network analysis have been found to be useful in many other fields, such as information theory and the study of transportation systems. The following analysis of traffic flow that was mentioned in the introduction illustrates how systems of linear equations with many solutions can arise in practice.

Consider the typical road network of following Figure. It represents an area of downtown Jacksonville, Florida. The streets are all one-way with the arrows indicating the direction of traffic flow. The traffic is measured in vehicles per hour (vph). The figures in and out of the network given here are based on midweek peak traffic hours, 7 A.M. to 9 A.M. and 4 P.M. to 6 P.M. Let us construct a mathematical model that can be used to analyse the flow $x_1, \ldots, x_4$ within the network.

Assume that the following traffic law applies. i.e., All traffic entering an intersection must leave that intersection.

This conservation of flow constraint (compare it to the first of Kirchhoff's laws for electrical networks) leads to a system of linear equations. These are, by intersection:

A: Traffic in $= x_1 + x_2$. Traffic out $= 400 + 225$. Thus $x_1 + x_2 = 625$.
B: Traffic in $= 350 + 125$. Traffic out $= x_1 + x_4$. Thus $x_1 + x_4 = 475$.
C: Traffic in $= x_3 + x_4$. Traffic out $= 600 + 300$. Thus $x_3 + x_4 = 900$.
D: Traffic in $= 800 + 250$. Traffic out $= x_2 + x_3$. Thus $x_2 + x_3 = 1050$.



The constraints on the traffic are described by the following system of linear equations.

$$\begin{aligned} x_1 + x_2 &= 625 \\ x_1 + x_4 &= 475 \\ x_3 + x_4 &= 900 \\ x_2 + x_3 &= 1050 \end{aligned}$$

The method of Gauss elimination is used to solve this system of equations.The augmented matrix and reduced echelon form of the preceding system are as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 625 \\ 1 & 0 & 0 & 1 & 475 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 1 & 1 & 0 & 1050 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations that corresponds to this reduced echelon form is

$$\begin{aligned} x_1 + x_4 &= 475 \\ x_2 - x_4 &= 150 \\ x_3 + x_4 &= 900 \end{aligned}$$

Expressing each leading variable in terms of the remaining variable, we get

$$\begin{aligned} x_1 &= -x_4 + 475 \\ x_2 &= x_4 + 150 \\ x_3 &= -x_4 + 900 \end{aligned}$$

As was perhaps to be expected, the system of equations has many solutions-there are many traffic flows possible. One does have a certain amount of choice at intersections. Let us now use this mathematical model to arrive at information. Suppose it becomes necessary to perform road work on the stretch DC of Monroe Street. It is desirable to have as small a flow x_3 as possible along this stretch of road. The flows can be controlled along various branches by means of traffic lights. What is the minimum value of x_3 along DC that would not lead to traffic congestion? We use the preceding system of equations to answer this question.

All traffic flows must be nonnegative (a negative flow would be interpreted as traffic moving in the wrong direction on a one-way street). The third equation tells us that x_3 will be a minimum when x_4 is as large as possible, as long as it does not go above 900. The largest value x_4 can be without causing negative values of x_1 or x_2 is 475. Thus, the smallest value of x_3 is $-475 + 900$, or 425. Any road work on Monroe should allow for at least 425vph.

In practice, networks are much vaster than the one discussed here, leading to larger systems of linear equations that are handled on computers. Various values of variables can be fed in and different scenarios created.

In general, we can construct the system a linear system of simultaneous equations of m equations in n unknowns can be expressed as

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m.$$

The above system in the matrix equivalent form can be expressed as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}_{m\times n}$$ is called the coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$ is called the matrix of unknowns and $$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$ is column matrix of constants.

If all b_i's for $i = 1 \cdots m$ are zero i.e., $b_1 = b_2 = \cdots = b_m = 0$, then the system is said to be homogenous and is said to be non-homogeneous if at least one b_i is non-zero.

AUGMENTED MATRIX:

Suppose we form a matrix of the form [A : B] by appending to A an extra column whose elements are columns of B i.e.,

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} : b_1 \\ a_{21} & a_{22} & . & . & . & a_{2n} : b_2 \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} : b_m \end{bmatrix}$$

is called the augmented matrix associated with the system and is denoted by $[A|B]$ or [A : B].

ELEMENTARY ROW TRANSFORMATIONS:

These are operations that are carried out on the rows of a given matrix. The following operations constitute the three row transformations.

- Interchange of i^{th} and j^{th} rows: $R_i \leftrightarrow R_j$.
- Multiplying each element of the i^{th} row by a non-zero constant k: $R_i' \to kR_i$
- Adding a constant k multiple of j^{th} row to i^{th} row: $R_i' \to R_i + kR_j$.

EQUIVALENT MATRICES:

Two matrices A and B are said to be **equivalent** if one can be obtained from the other by a sequence of Elementary transformation. Equivalent matrices are denoted by $\boldsymbol{A \sim B}$.

NOTE: All the above operations can also be performed on columns.

ECHELON FORM OR ROW ECHELON FORM:

A non-zero matrix A is said to be in echelon form, if

- The leading entry (non-zero element) of each non-zero row after the first row occurs to the right of the leading entry of the previous row.
- All the entries of a column below a leading entry are zero.
- All zero rows (all elements are zero) are at the bottom of the matrix.

A non-zero matrix A is an echelon matrix, if the number of zeros preceding the first non-zero entry of a row increases row by row until zero rows remaining.

RANK OF A MATRIX:

Rank of a matrix B is the number of non-zero rows in the row Echelon Form and is denoted by $\rho(B)$.

Example:

$B = \begin{bmatrix} 1 & 3 & 1 & 5 & 0 \\ 0 & 1 & 5 & 1 & 5 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row-echelon form.

The rank of an echelon matrix is the number of non-zero rows in it. i.e., $\rho(B) = 3$.

Problems

1. Given matrix Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

.

Solution: The rank of the matrix can be obtained by reducing it to row echelon form.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Perform $R_1 \leftrightarrow R_2$ i.e., interchanging row 1 and row 2 we get

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1,\ R_3' \to R_3 - 3R_1,\ R_4' \to R_4 - 6R_1$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$R_3' \to 5R_3 - 4R_2,\ R_4' \to 5R_4 - 9R_2$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$$R_4' \to R_4 - R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As there are no elements below the fourth diagonal element the process is complete.

$\rho(A) =$ Rank of $A =$ number of non-zero rows= 3.

2. Reduce the following matrix to echelon form and hence find its rank.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Solution: Given matrix is

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

interchanging row 1 and row 2 we get

Perform $R_3' \to R_3 - 2R_1, R_4' \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{bmatrix}$$

Perform $R_3' \to R_3 + R_2, R_4' \to R_4 + R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

Perform $R_4' \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Above matrix is in the echelon from, therefore rank of matrix A, $\rho(A) = 3$ (no. of non-zero rows).

3. Using the elementary transformations find the rank of the matrix

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

Solution: Given matrix is

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

$$R_2' \to R_2 + 2R_1,\ R_3' \to R_3 + 3R_1,\ R_4' \to R_4 + 5R_1$$

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$R_3' \to R_3 - 2R_2$, $R_4' \to R_4 - 2R_2$, we get

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix $B = 2$.

EXERCISE

Find the rank of the following matrices by reducing it to echelon form:

1. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

8. $\begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$

9. $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

10. $\begin{bmatrix} 221 & 22 & 23 & 24 \\ 22 & 23 & 24 & 25 \\ 23 & 24 & 25 & 26 \\ 24 & 25 & 26 & 27 \end{bmatrix}$

APPLICATIONS:

- One useful application of calculating the rank of a matrix is the computation of the number of solutions of a system of linear equations.
- In the classification of an image.
- If we view a matrix as a matrix of linear transformation, the rank talks about the dimension of the range.
- In control theory, the rank of a matrix can be used to determine whether a linear system is controllable, or observable.
- In the field of communication complexity, the rank of the communication matrix of a function gives bounds on the amount of communication needed for two parties to compute the function.

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS:

A system of linear equations such as (1) may or may not have a solution. However, existence of solution is guaranteed only if the system is homogeneous.

SOLUTION OF NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS:

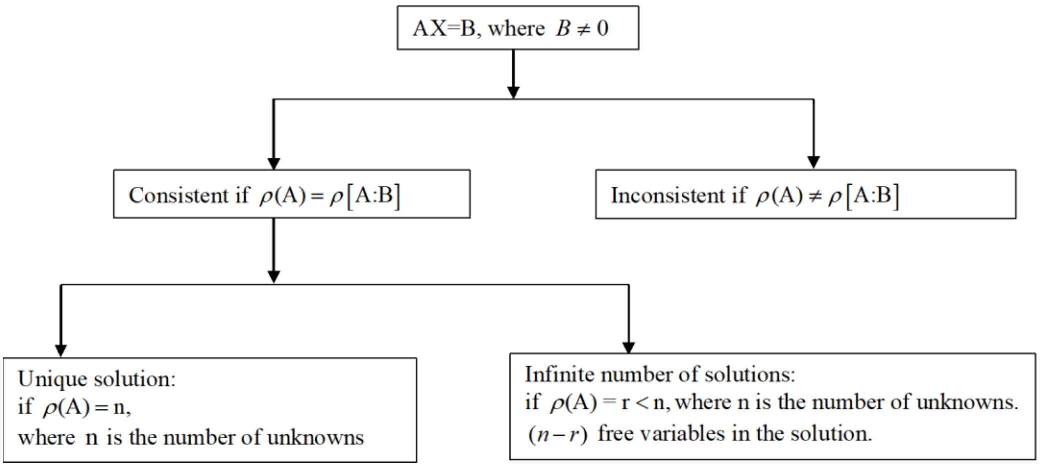
A non-homogeneous system of equations $\boldsymbol{AX} = \boldsymbol{B}$ is consistent if $\boldsymbol{r}$, the rank of coefficient matrix A is equal to $\boldsymbol{r'}$, the rank of the augmented matrix $[\mathrm{A} : \mathrm{B}]$ and has unique solution if $\boldsymbol{r} = \boldsymbol{r'} = \boldsymbol{n}$, the number of unknowns. If $\boldsymbol{r} = \boldsymbol{r'} < \boldsymbol{n}$ then the system possesses infinite number of solutions. The system is inconsistent if $\boldsymbol{r} \neq \boldsymbol{r'}$.

SOLUTION OF HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS:

A homogeneous system of linear equations $\boldsymbol{AX} = \boldsymbol{0}$ is always consistent as for such a system, $A = [\mathrm{A} : 0]$ and hence rank of coefficient matrix is equal to the rank of the augmented matrix. If rank of A is equal to the number of unknowns $\boldsymbol{n}$, the system has trivial solution i.e., all unknowns $x_1, x_2, \cdots, x_n$ are zero. A non-trivial solution exists to a system if and only if $|\boldsymbol{A}| = \boldsymbol{0}$ and hence the system has infinite number of solutions.

The following block diagram illustrates connection between rank of a matrix and consistence of that system.

NON-HOMOGENOUS SYSTEM



HOMOGENOUS SYSTEM

AX = B, where B = 0

↓

Always consistent

↓

Unique solution (trivial): X = 0
if $\rho(A) = n$, the number of unknowns.

Infinite number of solutions:
if $\rho(A) = r < n$, the number of unknowns,
then $(n-r)$ free variables in the solution.
If **A** is a square matrix, then $|\mathbf{A}| = \mathbf{0}$.

EXAMPLES

1. Test for consistency and solve

$$2x_1 - x_2 + 3x_3 = 1; -3x_1 + 4x_2 - 5x_3 = 0;\ x_1 + 3x_2 - 6x_3 = 0$$

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & :1 \\ -3 & 4 & -5 & :0 \\ 1 & 3 & -6 & :0 \end{bmatrix}$$

$$R_2' \to R_2 + (3/2)R_1,\ R_3' \to R_3 - (1/2)R_1$$

$$[A:B] \sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 7 & -15 & :-1 \end{bmatrix}$$

$$R_3' \to R_3 - (7/5)R_2$$

$$[A:B] \sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 0 & -68 & :-26 \end{bmatrix}$$

Hence, $\rho(A) = \rho([A:B]) = 3$ = number of unknowns.

Thus, the system of linear equations is consistent and possesses a unique solution.

To find the unknowns, consider the rows of [A : B] in the last step in terms of its equivalent equations ,

$$2x_1 - x_2 + 3x_3 = 1; \quad 5x_2 - x_3 = 3; \quad -68x_3 = -26.$$

Here we make use of **back substitution** in order to find the unknowns by considering, last equation to find x_3, next second to find x_2 and finally first equation to find x_1. From last equation we obtain x_3:

$$\text{i.e., } -68x_3 = -26 \Rightarrow x_3 = \frac{13}{34}.$$

Next, from second equation with substitution of x_3, we find x_2:

$$\text{i.e., } 5x_2 - x_3 = 3 \Rightarrow x_2 = \frac{3+x_3}{5} = \frac{3+\frac{13}{34}}{5} \Rightarrow x_2 = \frac{23}{34}.$$

Finally, to find the x_1 we make use first equation

$$\text{i.e., } 2x_1 - x_2 + 3x_3 = 1 \Rightarrow x_1 = \frac{1}{2}(1 + x_2 - 3x_3) = \frac{1}{2}\left(1 + \frac{23}{34} - 3\frac{13}{34}\right) \Rightarrow x_1 = \frac{9}{34}.$$

Thus, the unique solution is given by

$$x_1 = \frac{9}{34}, x_2 = \frac{23}{34}, x_3 = \frac{13}{34}.$$

2. Check the following system of equations for consistency and solve, if consistent.

$$x + 2y + 2z = 1, \quad 2x + y + z = 2, \quad 3x + 2y + 2z = 3, y + z = 0.$$

Solution : The augmented matrix is given by

$$[A{:}B] = \begin{bmatrix} 1 & 2 & 2 & :1 \\ 2 & 1 & 1 & :2 \\ 3 & 2 & 2 & :3 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1, R_3' \to R_3 - 3R_1$$

$$[A{:}B] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & -3 & -3 & :0 \\ 0 & -4 & -4 & :0 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_2' \to (-1/3)\, R_2, R_3' \to (-1/4)\, R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & 1 & 1 & :0 \\ 0 & 1 & 1 & :0 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_3' \to R_3 - R_2, R_4' \to R_4 - R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & 1 & 1 & :0 \\ 0 & 0 & 0 & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

$\rho(A) = \rho([A:B]) = 2 < 3$, number of unknowns.

Thus, the given system is consistent and possesses infinite number of solutions by assigning arbitrary values to $(n - r) = 3 - 2 = 1$, free variable.

$$\Rightarrow \quad x + 2y + 2z = 0$$
$$y + z = 0.$$

Here there are three unknowns, we should take z as the free variable and let $z = k$ (arbitrarily value). From second equation, $y + z = 0 \Rightarrow y = -z = -k$.

Finally, from first equation, $x + 2y + 2z = 1 \Rightarrow x = 1 - 2y - 2z = 1 - 2(-k) - 2k$

$$\Rightarrow x = 1.$$

Therefore, the solutions are given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ k \end{bmatrix}.$$

3. Show that the following system of equations is not consistent.

$$x + 2y + 3z = 6, 3x - y + z = 4, 2x + 2y - z = -3, -x + y + 2z = 5$$

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & :6 \\ 3 & -1 & 1 & :4 \\ 2 & 2 & -1 & :-3 \\ -1 & 1 & 2 & :5 \end{bmatrix}$$

$$R_2' \to R_2 - 3R_1, R_3' \to R_3 - 2R_1, R_4' \to R_4 + R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : 6 \\ 0 & -7 & -8 & :-14 \\ 0 & -2 & -7 & :-15 \\ 0 & 3 & 5 & : 11 \end{bmatrix}$$

$$R_3' \to R_3 - (2/7)R_2, R_4' \to R_4 + (3/7)R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : 6 \\ 0 & -7 & -8 & :-14 \\ 0 & 0 & -33 & :-77 \\ 0 & 0 & 11 & : 35 \end{bmatrix}$$

$$R_4' \to R_4 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : 6 \\ 0 & -7 & -8 & :-14 \\ 0 & 0 & -33 & :-77 \\ 0 & 0 & 0 & : 28 \end{bmatrix}$$

$$\rho(A) = 3 \text{ and } \rho([A:B]) = 4$$

$\rho(A) \neq \rho([A:B])$.

Therefore, the given system is inconsistent and it has no solution.

4. Check the following system of equations for consistency and solve, if consistent.

$$x + y - 2z = 3, 2x - 3y + z = -4, 3x - 2y - z = -1, y - z = 2.$$

Solution: Consider the augmented matrix,

$$[A:B] = \begin{bmatrix} 1 & 1 & -2 & : 3 \\ 2 & -3 & 1 & :-4 \\ 3 & -2 & -1 & :-1 \\ 0 & 1 & -1 & : 2 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1, R_3' \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & : 3 \\ 0 & -5 & 5 & :-10 \\ 0 & -5 & 5 & :-10 \\ 0 & 1 & -1 & : 2 \end{bmatrix}$$

$$R_3' \to R_3 - R_2, \quad R_4' \to R_4 + (1/5)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & : 3 \\ 0 & -5 & 5 & :-10 \\ 0 & 0 & 0 & : 0 \\ 0 & 0 & 0 & : 0 \end{bmatrix}$$

We see that $\rho(A) = \rho([A : B]) = 2 < 3,$ number of unknowns.

Thus, the equations are consistent and possess infinite number of solutions with

$(n-r) = 3-2 = 1,$ free variable.

The corresponding equations are $x+y-2z=3$ and $-5y+5z=-10$.

Let us choose $z=k$ (arbitrary constant). Then from second equation we have

$$-5y+5z=-10 \Rightarrow y = -\tfrac{1}{5}(-10-5z) = -\tfrac{1}{5}(-10-5k) = 2+k.$$

From first equation $x+y-2z=3 \Rightarrow x = 3-y+2z = 3-(2+k)+2k \Rightarrow x = 1+k.$

Therefore, the solution is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+k \\ 2+k \\ k \end{bmatrix}.$$

5. Find the values of λ for which the system $x+y+z=1,\quad x+2y+4z=\lambda,$
   $x+4y+10z=\lambda^2$, has a solution. Solve it in each case.

Solution: The augmented matrix is given by

$$[\text{A:B}] = \begin{bmatrix} 1 & 1 & 1 & :1 \\ 1 & 2 & 4 & :\lambda \\ 1 & 4 & 10 & :\lambda^2 \end{bmatrix}$$

$$R_2' \to R_2 - R_1, \text{R}_3' \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 3 & 9 & : & \lambda^2-1 \end{bmatrix}$$

$$R_3' \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 0 & 0 & : & \lambda^2-3\lambda+2 \end{bmatrix}$$

We observe that $\rho(A)=2$ and $\rho([\text{A : B}])$will be equal to 2 iff $\lambda^2-3\lambda+2=0$.

i.e., for $\lambda=1$ or $\lambda=2$.

$\Rightarrow$ System will possess a solution if $\lambda = 1$or 2 and in both the cases the system will have infinite number of solution as $\rho(A) = \rho([\text{A : B}]) = 2 < 3,$ number of unknowns and hence 1 free variable. Let us consider these cases one by one.

<u>Case 1:</u> When $\lambda = 1$, the reduced system gives

$$x+y+z=1$$

$$y+3z=1-1=0.$$

Let $z=k$ be arbitrary and from second equation we have

$$y=-3z=-3k.$$

From first equation, we have

$$x = 1 - y - z = 1 - (-3k) - k = 1 + 2k.$$

Case 2: When $\lambda = 2$, the reduced system gives,

$$x + y + z = 1 \text{ and } \; y + 3z = 2 - 1 = 1.$$

Let $z = k$, then $y = 1 - 3k$ and $x = 1 - y - z = 1 - 1 + 3k - k = 2k$, where k is an arbitrary constant.

6. Find the values of $\lambda$and μ for which the system $x + y + z = 6, x + 2y + 3z = 10,$

$x + 2y + z = \mu$ has (i) a unique solution (ii) infinitely many solutions (iii) no solution.

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & :6 \\ 1 & 2 & 3 & :10 \\ 1 & 2 & \lambda & :\mu \end{bmatrix}$$

$$R_2' \to R_2 - R_1, R_3' \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$$R_3' \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}.$$

Here we observe that

a) If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ i.e., $\lambda = 3$and$\mu \neq 10$, then the system will be inconsistent and possesses no solution.

b) If $\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e., $\lambda = 3$ and $\mu = 10$ the system will reduce to
In this case the system possesses infinite solutions.

c) If $\lambda - 3 \neq 0$ i.e., $\lambda \neq 3$, the system will possess a unique solution, irrespective of the value of μ.

EXERCISE

1. Show that the system $x + y + z = 4;\ 2x + y - z = 1;\ x - 2y + 2z = 2$ is consistent and solve the system.
2. Find the value of **λ** for which the system has solution. Solve the system in each possible case: $x + y + z = 1;\ \ x + 2y + 4z = \lambda;\ \ x + 4y + 10z = \lambda^2$.
3. Test for consistency and solve:

   $a)$ $x + 2y + 2z = 5, 2x + y + 3z = 6, 3x - y + 2z = 4, x + y + z = -1$

   $b)$ $5x + 3y + 7z = 4,\ \ 3x + 26y + 2z = 9,\ \ 7x + 2y + 10z = 5.$

   $c)$ $5x + y + 3z = 20,\ \ 2x + 5y + 2z = 18,\ \ 3x + 2y + z = 14.$
4. Investigate the value of **λ** and **μ** so that the equations

   $2x + 3y + 5z = 9, \qquad 7x + 3y - 2z = 8, \qquad 2x + y + \lambda z = \mu,$

   have (i) no solution (ii) unique solution (iii) infinite solutions.
5. Find the values of **λ** and **μ** such that the system

   $x + 2y + 3z = 6, \qquad x + 3y + 5z = 9, \qquad 2x + 5y + \lambda z = \mu,$

   have (i) no solution (ii) unique solution (iii) infinite solutions.
6. For what values of **λ** and **μ** do the system of equations:

   $x + y + z = 6, \quad x + 2y + 5z = 10, \quad x + 3y + \lambda z = \mu$

   have (i) no solution (ii) unique solution (iii) infinite solutions.

GAUSS ELIMINATION METHOD:

The Gauss Elimination Method is a fundamental technique in linear algebra used to solve systems of linear equations.

Procedure:

- Write the augmented matrix of the given system of equations
- Reduce the augmented matrix to Echelon form
- Write the linear equations (starting from last) associated with the echelon form of matrix

EXERCISE

1. Solve the following system by Gauss elimination method

$$x + y - z = 0, 2x - 3y + z = -1, x + y + 3z = 12, y + z = 5$$

Solution : The augmented matrix is given by

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & :0 \\ 2 & -3 & 1 & :-1 \\ 1 & 1 & 3 & :12 \\ 0 & 1 & 1 & :5 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1, R_3' \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 1 & 1 & :1 \end{bmatrix}$$

$$R_4' \to R_4 + (1/5)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 0 & 8 & :24 \end{bmatrix}$$

$$R_4' \to R_4 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

By back substitution

$$4z = 12 \Rightarrow z = 3,$$
$$5y + 3z = -1 \Rightarrow y = 2,$$
$$x + y - z = 0 \Rightarrow x = 2.$$

2. Solve the following system by Gauss elimination method

$$2x_1 - x_2 + 2x_3 = 1$$
$$-3x_1 + 4x_2 - 5x_3 = 0$$
$$x_1 + 3x_2 - 6x_3 = 0.$$

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & :1 \\ -3 & 4 & -5 & :0 \\ 1 & 3 & -6 & :0 \end{bmatrix}$$

$$R_2' \to R_2 + (3/2)R_1, R_3' \to R_3 - (1/2)R_1$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 7 & -15 & :-1 \end{bmatrix}$$

$$R_3' \to R_3 - (7/5)R_2$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 0 & -68 & :-26 \end{bmatrix}$$

$$2x_1 - x_2 + 3x_3 = 1$$
$$\Rightarrow 5x_2 - x_3 = 3$$
$$-68x_3 = -26.$$

By back substitution the solution is given by

$$x_3 = \frac{13}{34}, x_2 = \frac{23}{34}, x_1 = \frac{9}{34}.$$

3. For the Example 2 (Mentioned in page 4-5), by using the method of Gauss-elimination, we get

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 8 \\ 0 & 4 & 1 & 16 \end{bmatrix}$$

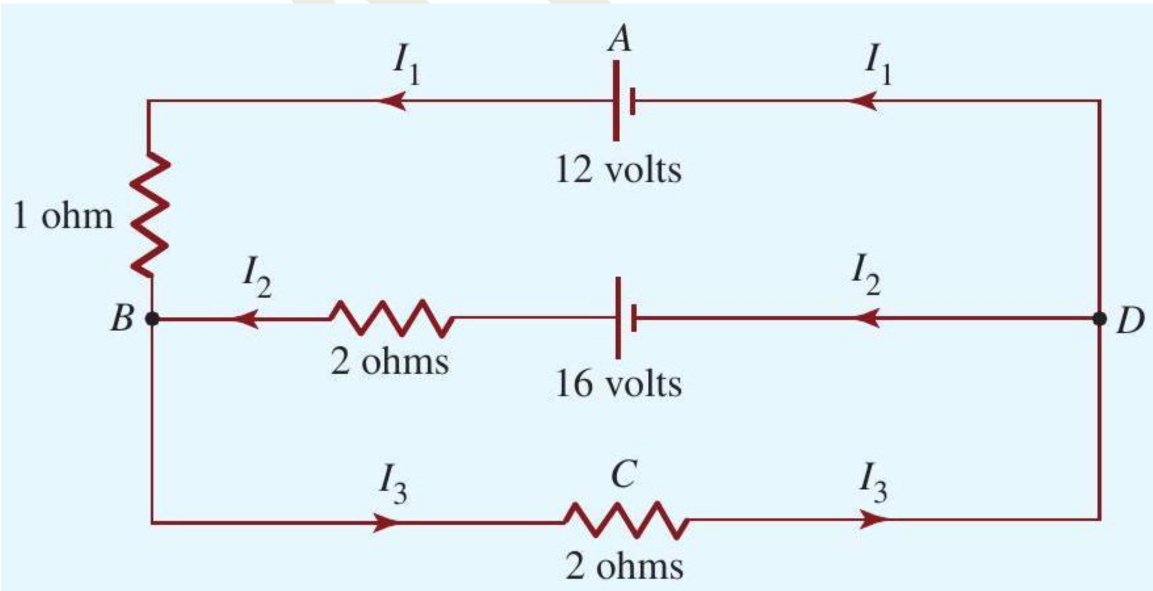
$$\approx R2 + (-4)R1 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 4 & 1 & 16 \end{bmatrix}$$

$$\underset{\left(-\frac{1}{4}\right)R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 4 & 1 & 16 \end{bmatrix} \approx \begin{matrix} R3 + (-4)R2 \\ R1+(-1)R2 \\ 0 \end{matrix} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 2 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 0 & 6 & 24 \end{bmatrix}$$

$$\underset{\left(\frac{1}{6}\right)R3}{\approx} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 2 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Then by back substitution, the currents are $I_1 = 1,\ I_2 = 3,\ I_3 = 4$. The units are amps. The solution is unique, as is to be expected in this physical situation.

4. Determine the currents through the various branches of the electrical network. This example illustrates how one has to be conscious of direction in applying Law 2 for closed paths.



Junctions:

$$\text{Junction } B, \quad I_1 + I_2 = I_3$$
$$\text{Junction } D, \quad I_3 = I_1 + I_2$$

giving

$$I_1 + I_2 - I_3 = 0.$$

Paths:

$$\text{Path } ABCDA, \quad 1I_1 + 2I_3 = 12$$
$$\text{Path } ABDA, \quad 1I_1 + 2(-I_2) = 12 + (-16)$$

Observe that we have selected the direction $ABDA$ around this last path. The current along the branch BD in this direction is $-I_2$, and the voltage is -16. We now have three equations in the three variables I_1, I_2, and I_3.

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ I_1 + 2I_3 &= 12 \\ I_1 - 2I_2 &= -4 \end{aligned}$$

Solving these equations by Gauss-elimination method, we get $I_1 = 2, I_2 = 3, I_3 = 5$ amps.

In practice, electrical networks can involve many resistances and circuits; determining currents through branches involves solving large systems of equations on a computer.

Note: In circuit analysis, when solving equations using Kirchhoff's laws or Ohm's law, if the calculated current is negative, it indicates that the direction of flow is opposite to the assumed direction. It doesn't mean there's a negative quantity of charge flowing; it simply signifies a direction opposite to the assumed convention.

For example, if you assume a certain direction for the current in a circuit, and after calculations, you find the current to be -2 amperes, it means the actual direction of current is opposite to your assumed direction, and the magnitude is 2 amperes.

So, the negative sign in the context of current in an electric circuit is a mathematical representation of the direction of electron flow, not an indication of negative charge.

5**.** John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = amount\ invested\ in\ money\ market\ fund$$
$$y = amount\ invested\ in\ municipal\ bonds$$
$$z = amount\ invested\ in\ mutual\ funds$$

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x + y + z = 12{,}000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z = y + 4{,}000$$

The third equation shows that the total amount of interest earned from each fund equals $670.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$x + y + z = 12{,}000$$
$$-y + z = 4{,}000$$
$$0.03x + 0.04y + 0.07z = 670$$

By solving the above system with Gauss elimination, we obtain John invested $2,000 in a money-market fund, $3,000 in municipal bonds, and $7,000 in mutual funds.

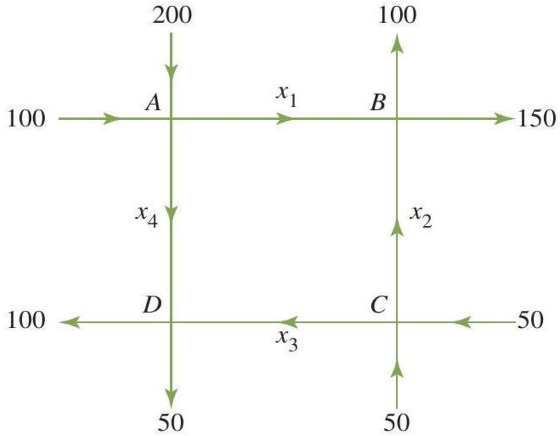
EXERCISE

Solve the following system of equations by Gauss Elimination method:

1. $x + 2y + z = 3,\ \ 2x + 3y + 2z = 5,\ \ 3x - 5y + 5z = 2$
2. $x + y + z = 9,\ \ \ x - 2y + 3z = 8,\ 2x + y - z = 3.$
3. $2x - 3y + 4z = 7,\ \ 5x - 2y + 2z = 7,\ \ 6x - 3y + 10z = 23.$
4. $2x + y + 4z = 12,\ \ 4x + 11y - z = 33,\ \ 8x - 3y + 2z = 20.$
5. $4x + y + z = 4,\ \ x + 4y - 2z = 4,\ \ 3x + 2y - 4z = 6.$
6. $3x - y + 2z = 12,\ \ x + 2y + 3z = 11,\ 2x - 2y - z = 2$
7. $2x - y + 3z = 1,\ \ -3x + 4y - 5z = 0,\ \ x + 3y + 6z = 0.$
8. $2x + 5y + 7z = 52,\ \ 2x + y - z = 0,\ \ x + y + z = 9.$
9. $x - 2y + 3z = 2,\ \ 3x - y + 4z = 4,\ \ 2x + y - 2z = 5.$
10. $5x_1 + x_2 + x_3 + x_4 = 4,\ x_1 + 7x_2 + x_3 + x_4 = 12,\ \ x_1 + x_2 + 6x_3 + x_4 = -5,$

    $x_1 + x_2 + x_3 + 4x_4 = -6.$

11. Construct a system of linear equations that describes the traffic flow in the road network of Figure. All streets are one-way streets in the directions indicated. The units are vehicles per hour. Give two distinct possible flows of traffic. What is the minimum possible flow that can be expected along branch AB ?



12. Figure 1.32 represents the traffic entering and leaving a "roundabout" road junction. Such junctions are very common in Europe. Construct a system of equations that describes the flow of traffic along the various branches. What is the minimum flow possible along the branch BC ? What are the other flows at that time? (Units are vehicles per hour.)

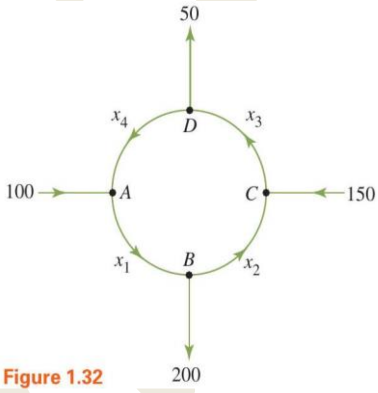
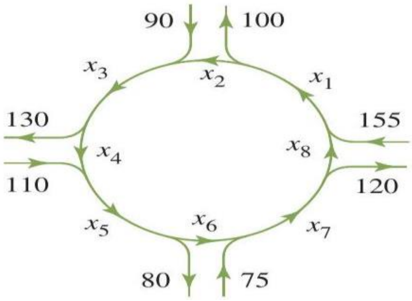
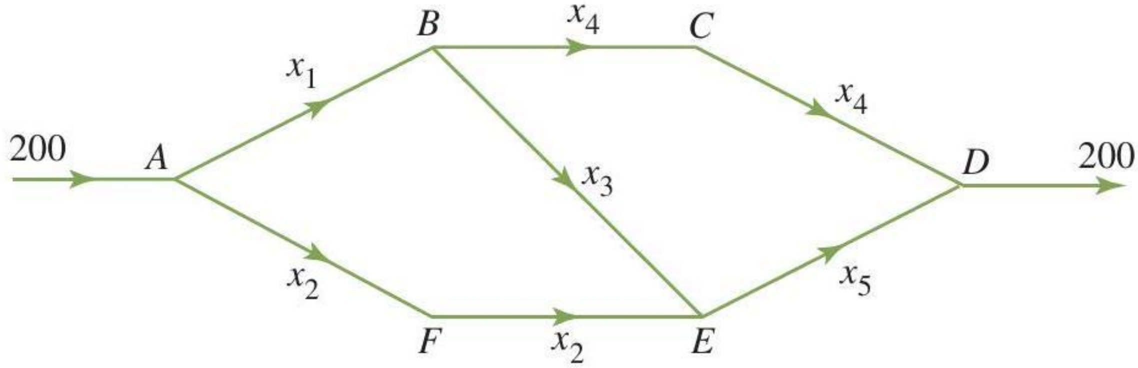


Figure 1.32

13. Figure represents the traffic entering and leaving another type of roundabout road junction in Continental Europe. Such roundabouts ensure the continuous smooth flow of traffic at road junctions. Construct linear equations that describe the flow of traffic along the various branches. Use these equations to determine the minimum flow possible along x_1. What are the other flows at that time? (It is not necessary to compute the reduced echelon form. Use the fact that traffic flow cannot be negative.)



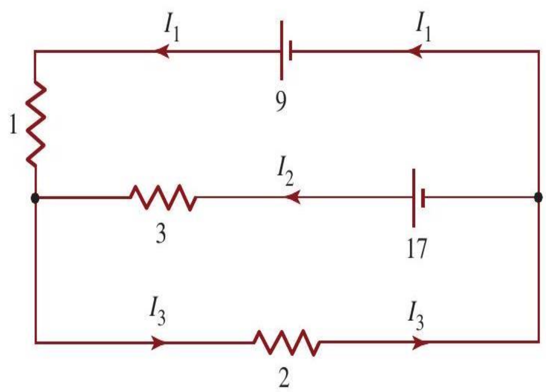
14. Figure describes a flow of traffic, the units being vehicles per hour.

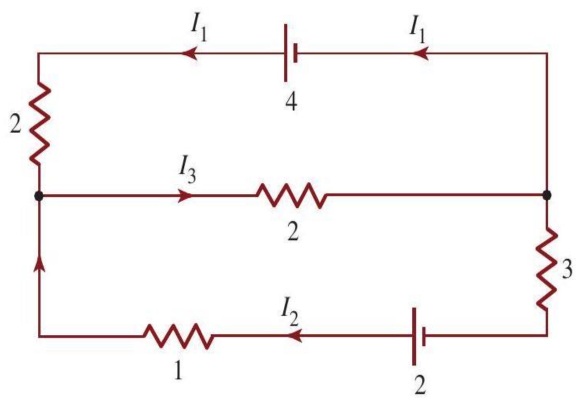


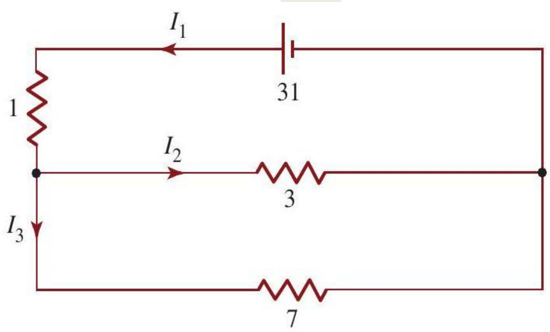
(a) Construct a system of linear equations that describes this flow.

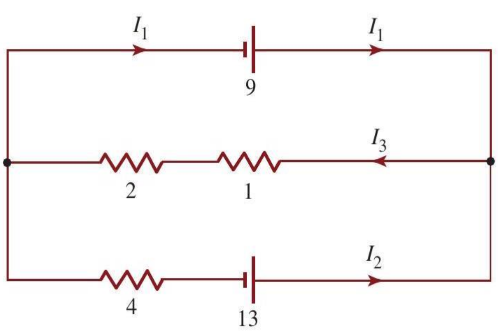
(b) The total time it takes the vehicles to travel any stretch of road is proportional to the traffic along that stretch. For example, the total time it takes x_1 vehicles to traverse AB is kx_1 minutes. Assuming that the constant is the same for all sections of road, the total time for all these 200 vehicles to be in this network is $kx_1 + 2kx_2 + kx_3 + 2kx_4 + kx_5$. What is this total time if $k = 4$? Give the average time for each car.

15. **Determine the currents in the various branches of the electrical networks. The units of current are amps and the units of resistance are ohms.**









16. Suppose Raj has $18,000 to invest. He invests a portion of the money in cryptocurrency with a 12% annual return and the rest in real estate with a 6% annual return. After one year, he earns a total income of $2,160. Determine how much Raj invested in each category.

17. Emily received a prize of $15,000, which she decided to divide into three parts and invest in three different ways: in a savings account paying 2% annual interest, in corporate bonds paying 5% annual interest, and in technology stocks paying 9% annual interest. Emily invested $3,000 more in technology stocks than in the savings account. She earned $850 in interest the first year. Determine how much Emily invested in each type of fund.

Solution: savings=2000, corporate bond=8000, technology stock=5000 dollars

DIAGONALLY DOMINANT FORM:

A system of 'n' linear equations in 'n' unknowns given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2 \quad (2)$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b_n \quad (n)$$

is said to be in diagonally dominant form if in equation (1), $|a_{11}|$is greater the sum of the absolute values of the remaining coefficients; in (2), $|a_{22}|$ is greater than the sum of the absolute of the remaining coefficients and so on.

$$|a_{11}| > |a_{12}| + |a_{13}| + \ldots\ldots\ldots |a_{1n}|$$

$$|a_{22}| > |a_{21}| + |a_{23}| + \ldots\ldots\ldots |a_{2n}|$$

$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \ldots\ldots\ldots |a_{n(n-1)}|.$$

GAUSS-SEIDEL METHOD:

The Gauss-Seidel method is an iterative method that can be used to solve a system of 'n' linear equations in 'n' unknowns. A starting or an initial solution is first assumed, which is then improved through successive iteration. A convergence to the actual solution is ensured if the given system of equations is arranged in the diagonally dominant form. The following example illustrates the working procedure of this method.

EXAMPLES

1. Solve the following system of equations using Gauss-Seidel method.

$$6x + 15y + 2z = 72, x + y + 54z = 110, 27x + 6y - z = 85.$$

Solution: In the above equations, we have

$|15| > |6| + |2|, |54| > |1| + |1| \; \& \; |27| > |6| + |-1|$.

Hence the equations are arranged in the diagonally dominant form as:

$27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110.$

The first equation is used to determine *x* and is therefore rewritten as

$x = \frac{85-6y+z}{27}$. (1)

The second equation is used to determine *y* and is rewritten as

$y = \frac{72-6x-2z}{15}$. (2)

The third equation used to determine *z* is rearranged as

$z = \frac{110-x-y}{54}$. (3)

Equations (1), (2), (3) are used to find sequentially *x*, *y* and *z* in each of the iterations.

Starting solution: Let us choose $[x, y, z] = [0,0,0]$ as the starting solution.

First iteration:

$$x^{(1)} = \frac{1}{27}[85 - 0 + 0] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72 - 6(3.1481) - 0] = 3.5407$$

$z^{(1)} = \frac{1}{54}[110 - 3.1481 - 3.5407] = 1.9132.$

Note that in finding $y^{(1)}$ the latest value $x^{(1)} = 3.1481$ is used and not x=0. Similarly, in finding $z^{(1)}$, the latest values $y^{(1)} = 3.5407$.

Second iteration:

$$x^{(2)} = \frac{1}{27}[85 - 6(3.5407) + 1.9132] = 2.4322$$

$$y^{(2)} = \frac{1}{15}[72 - 6(2.4322) - 2(1.9132] = 3.5720$$

$$z^{(2)} = \frac{1}{54}[110 - 2.4322 - 3.5720] = 1.9258.$$

and$x^{(1)} = 3.1481$ are used. The same procedure is applied in subsequent iterations also.

Third iteration:

$$x^{(3)} = \frac{1}{27}[85 - 6(3.5720) + 1.9258] = 2.4257$$

$$y^{(3)} = \frac{1}{15}[72 - 6(2.4257) - 2(1.9258)] = 3.5729$$

$$z^{(3)} = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.9259.$$

Therefore $[x, y, z] = [2.4257, 3.5729, 1.9259]$.

Fourth iteration:

$$x^{(4)} = \frac{1}{27}[85 - 6(3.5729) + 1.9259] = 2.4255$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.4255) - 2(1.9259)] = 3.5730$$

$$z^{(4)} = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9259.$$

Since the solutions in 3^{rd} and 4^{th} iterations agree upto 3 places of decimals, the solution can be taken as

$[x, y, z] = [2.4255, 3.5730, 1.9259]$.

EXERCISE

Solve the following system of equations by Gauss Seidel Method performing 3 iterations:

1. $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.
2. $3x + 8y + 29z = 71$, $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$.
3. $10x + 2y + z = 9$, $x + 10y - z = -22, -2x + 3y + 10z = 22$.

4. $5x - y = 9$, $x - 5y + z = -4$, $y - 5z = 6$ taking $\left(\frac{9}{5}, \frac{4}{5}, \frac{6}{5}\right)$ as first approximation.

5. $x + y + 54z = 110$, $27x + 6y - z = 85$, $6x + 15y + 2z = 72$.

6. $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$

7. $12x + y + z = 31$, $2x + 8y - z = 24$, $3x + 4y + 10z = 58$.

8. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$.

9. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$ taking initial approximation as (1, 0, 3).

10. $10x + 2y + z = 9$, $2x + 20y - 2z = -44$, $-2x + 3y + 10z = 22$ by taking (0, 0, 0) as initial approximation root (carry out 3 iterations).

Systems of Linear Differential Equations

Example:

Let's consider a mixture problem that involves two tanks with varying concentrations of a substance, and the goal is to model the transfer of the substance between the tanks using a system of linear differential equations.

Mixing Tanks with Saltwater:

Suppose there are two tanks, Tank A and Tank B. Tank A contains 200 liters of water with a salt concentration of 0.1 kg/L, and Tank B contains 150 liters of water with a salt concentration of 0.05 kg/L. The contents of the tanks are mixed continuously, and a pump transfers water from Tank A to Tank B at a rate of 2 liters per minute. Simultaneously, another pump transfers water from Tank B to Tank A at a rate of 1.5 liters per minute.

Let $x(t)$ represent the amount of salt (in kg) in Tank A at time (t), and $y(t)$ represent the amount of salt (in kg) in Tank B at time (t).

The rate of change of salt in Tank A is given by the differential equation:

$$\frac{dx}{dt} = -\frac{2}{200}x + \frac{1.5}{150}y$$

Similarly, the rate of change of salt in Tank B is given by:

$$\frac{dy}{dt} = \frac{2}{200}x - \frac{1.5}{150}y$$

In matrix form, the system can be represented as:

$$\frac{d\begin{bmatrix}x\\y\end{bmatrix}}{dt} = \begin{bmatrix}-\frac{2}{200} & \frac{1.5}{150}\\ \frac{2}{200} & -\frac{1.5}{150}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Here, the matrix on the right side (A) represents the coefficients of the system.

Solving a system of linear differential equations using the matrix method involves expressing the system in matrix form and then solving for the matrix of solutions. Let's consider a system of first-order linear differential equations as an example:

$$\frac{dx}{dt} = A.x$$

Here, (x) is a column vector of functions of (t), and (A) is a constant matrix. The solution to this system is given by:

$$x(t) = e^{At} \cdot x_0$$

Where (x_0) is the initial condition vector, and e^{At} is the matrix exponential.

Now, let's say you have a system of two first-order linear differential equations:

$$\frac{dx_1}{dt} = a_{\{11\}}x_1 + a_{\{12\}}x_2$$

$$\frac{dx_2}{dt} = a_{\{21\}}x_1 + a_{\{22\}}x_2$$

You can write this system in matrix form as:

$$\frac{d\begin{bmatrix}x_1\\x_2\end{bmatrix}}{dt} = \begin{bmatrix}a_{\{11\}} & a_{\{12\}}\\ a_{\{21\}} & a_{\{22\}}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Let (A) be the matrix on the right side and (X) be the column vector $[x_1,\ x_2]^T$. Then the system becomes:

$$\frac{dX}{dt} = A.X$$

Now, to solve this system, you can find the matrix exponential e^{At} and then multiply it by the initial condition vector (X_0) to obtain the solution ($X(t)$):

$$X(t) = e^{At} \cdot X_0$$

The matrix exponential ($e^{\{At\}}$) is often calculated using power series, and it involves eigenvalues and eigenvectors of the matrix (A). The general solution for e^{At} is given by:

$$e^{At} = P \cdot e^{Dt} \cdot P^{-1}$$

Here, (P) is the matrix of eigenvectors, and (D) is a diagonal matrix with eigenvalues on the diagonal.

EXAMPLE1:

Solve a system of linear differential equations using the matrix method.

Consider the system:

$$x_1' = -x_1 + 2x_2$$
$$x_2' = 2x_1 - x_2$$

You can write this system in matrix form as:

$$\frac{d\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{dt} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Calculate the eigenvalues λ and corresponding eigenvectors $\boldsymbol{v}$ of the matrix $\boldsymbol{A}$

$$\det(A - \lambda I) = 0$$

Solving this equation gives the eigenvalues:

$$\lambda_1 = 1, \ \lambda_2 = -3$$

For $\lambda_1 = 1$:

$$A - \lambda_1 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

Solving $(A - \lambda_1 I)\ \boldsymbol{v}_1 = 0$ gives the eigen vector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -3$:

$$A - \lambda_2 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Solving $(A - \lambda_2 I)\,\boldsymbol{v}_2 = 0$ gives the eigen vector $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$e^{At} = P \cdot e^{Dt} \cdot P^{-1}$$

$$e^{Dt} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The general solution is given by: $X(t) = e^{At}.X_0$

Where X_0 is the initial condition vector. If, for example $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then:

$$X(t) = \begin{bmatrix} \frac{1}{2}(e^{t} + e^{-3t}) & \frac{1}{2}(e^{t} - e^{-3t}) \\ \frac{1}{2}(e^{t} - e^{-3t}) & \frac{1}{2}(e^{t} + e^{-3t}) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This gives the solution for $x_1(t)$ and $x_2(t)$ in terms of t.

EXAMPLE 2: Solve a system of linear differential equations using the matrix method.

$$y_1' = 3y_1 + 4y_2$$
$$y_2' = 3y_1 + 2y_2$$

Solution: The coefficient matrix for the above system of ODE is $A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$

By solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we obtain the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -1$.

We see that $\mathbf{x}_1 = (4,3)^T$ is an eigenvector belonging to λ_1 and $\mathbf{x}_2 = (1,-1)^T$ is an eigenvector belonging to λ_2. Thus, any vector function of the form

$$\mathbf{Y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 = \begin{pmatrix} 4c_1 e^{6t} + c_2 e^{-t} \\ 3c_1 e^{6t} - c_2 e^{-t} \end{pmatrix}$$

is a solution of the system.
In case, suppose that we require that $y_1 = 6$ and $y_2 = 1$ when $t = 0$. Then

$$\mathbf{Y}(0) = \begin{pmatrix} 4c_1 + c_2 \\ 3c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

and it follows that $c_1 = 1$ and $c_2 = 2$. Hence, the solution of the initial value problem is given by

$$\mathbf{Y} = e^{6t}\mathbf{x}_1 + 2e^{-t}\mathbf{x}_2 = \begin{pmatrix} 4e^{6t} + 2e^{-t} \\ 3e^{6t} - 2e^{-t} \end{pmatrix}$$

3. $$\begin{aligned} y_1' &= -4y_1 - 6y_2 \\ y_2' &= 3y_1 + 5y_2 \end{aligned}$$

Solution: $y_1 = -c_1e^{2t} - 2c_2e^{-t}\,; y_2 = c_1e^{2t} + c_2e^{-t}$

4. $$\begin{aligned} y_1' &= 5y_1 + 4y_2 \\ y_2' &= y_1 + 2y_2 \end{aligned}$$

Solution: $y_1 = 4e^{6t} - c_2e^{t}\,; y_2 = c_1e^{6t} + c_2e^{t}$

Application on Mixtures

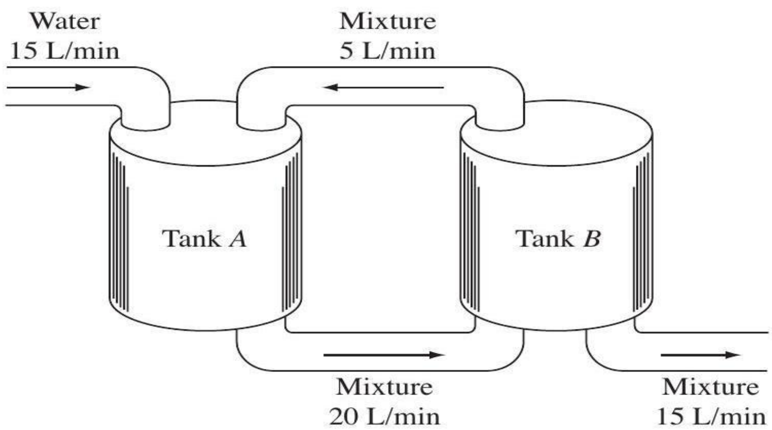
Two tanks are connected as shown in following Figure. Initially, tank A contains 200 liters of water in which 60 grams of salt has been dissolved and tank B contains 200 liters of pure water. Liquid is pumped in and out of the two tanks at rates shown in the diagram. Determine the amount of salt in each tank at time t.

Solution: Let $y_1(t)$ and $y_2(t)$ be the number of grams of salt in tanks A and B, respectively, at time t. Initially,

$$\mathbf{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

The total amount of liquid in each tank will remain at 200 liters, since the amount being pumped in equals the amount being pumped out. The rate of change in the amount of salt for each tank is equal to the rate at which it is being added minus the rate at which it is being pumped out. For tank A, the rate at which the salt is added is given by

$$(5\ \text{L/min}) \cdot \left(\frac{y_2(t)}{200}\ \text{g/L}\right) = \frac{y_2(t)}{40}\ \text{g/min}$$



and the rate at which the salt is being pumped out is

$$(20 \text{ L/min}) \cdot \left(\frac{y_1(t)}{200} \text{ g/L}\right) = \frac{y_1(t)}{10} \text{ g/min}$$

Thus, the rate of change for tank A is given by

$$y_1'(t) = \frac{y_2(t)}{40} - \frac{y_1(t)}{10}$$

Similarly, for tank B, the rate of change is given by

$$y_2'(t) = \frac{20y_1(t)}{200} - \frac{20y_2(t)}{200} = \frac{y_1(t)}{10} - \frac{y_2(t)}{10}$$

To determine $y_1(t)$ and $y_2(t)$, we must solve the initial value problem

$$\mathbf{Y}' = A\mathbf{Y},\ \mathbf{Y}(0) = \mathbf{Y}_0$$

where

$$A = \begin{pmatrix} -\frac{1}{10} & \frac{1}{40} \\ \frac{1}{10} & -\frac{1}{10} \end{pmatrix},\ \mathbf{Y}_0 = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = -\frac{3}{20}$ and $\lambda_2 = -\frac{1}{20}$, with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The solution must then be of the form

$$\mathbf{Y} = c_1 e^{-3t/20}\mathbf{x}_1 + c_2 e^{-t/20}\mathbf{x}_2$$

When $t = 0, \mathbf{Y} = \mathbf{Y}_0$. Thus,

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{Y}_0$$

and we can find c_1 and c_2 by solving

$$\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

The solution of this system is $c_1 = c_2 = 30$. Therefore, the solution of the initial value problem is

$$\mathbf{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 30e^{-3t/20} + 30e^{-t/20} \\ -60e^{-3t/20} + 60e^{-t/20} \end{pmatrix}.$$

EXERCISE

Solve the system of Linear Differential Equations and give the general solution of each of the following systems:

(a) $y_1 = y_1 + y_2$; $y_2 = -2y_1 + 4y_2$

Solution : $y_1 = c_1e^{2t} + c_2e^{3t}$; $y_2 = c_1e^{2t} + 2c_2e^{3t}$

(b) $y_1 = 2y_1 + 4y_2$; $y_2 = -y_1 - 3y_2$

Solution : $y_1 = -c_1e^{-2t} - 4c_2e^{t}$; $y_2 = c_1e^{-2t} + c_2e^{t}$

(c) $y_1 = y_1 - 2y_2$; $y_2 = -2y_1 + 4y_2$

Solution : $y_1 = 2c_1 + c_2e^{5t}$; $y_2 = c_1 - 2c_2e^{5t}$

Solve each of the following initial value problems:

(a) $y_1 = -y_1 + 2y_2$; $y_2 = 2y_1 - y_2$; $y_1(0) = 3$, $y_2(0) = 1$

(b) $y_1 = y_1 - 2y_2$; $y_2 = 2y_1 + y_2$; $y_1(0) = 1$, $y_2(0) = -2$

Module Outcomes:

At the end of the course, the student will be able to:

- Illustrate the knowledge of fundamental concepts of Linear algebra.
- Apply suitable techniques to solve given engineering and scientific problems related to Linear algebra based on the acquired knowledge.
- Analyze mathematical solutions of engineering and scientific problems related to Linear algebra and predict their behavior in real-world scenario.
- Use MATLAB to perform mathematical computation related to Linear algebra.