# MODULE -4:

# VECTOR SPACE

## CONTENTS:

**Vector spaces:**

1. Definition and Examples,
2. Subspace,
3. Linearly Independent and Dependent Sets,
4. Linear Span,
5. Basis and Dimension,
6. Inner product spaces,
7. Projection and orthogonality.

**(RBT Levels: L1, L2 & L3)**

### LEARNING OBJECTIVES:

Students will be able to understand and apply the concept of vector spaces by demonstrating proficiency in defining vector spaces and providing examples, identifying subspaces, recognizing linearly independent and dependent sets, determining linear spans, grasping the concepts of basis and dimension, comprehending inner product spaces, and applying knowledge of projection and orthogonality.

## LAB COMPONENT:

Computation of inner product and orthogonality for a vector space using MATLAB

## MODULE OUTCOMES:

After Completion of this module, student will be able to:

- Understand the formal definition of a vector space and its key properties.
- Apply linear span to solve problems related to spanning sets, dependence, and independence of vectors.
- Study Eigen values and Eigen vectors of square matrices, along with changing coordinates from one basis to another.
- Lay a strong foundation to perform computations of the learned mathematical concepts using MATLAB.

## APPLICATIONS OF VECTOR SPACE:

Vector spaces have wide applications across various fields due to their flexibility in modeling and analyzing different phenomena. Here are a few important applications:

**1. Physics and Engineering:** In physics for describing quantities like forces, velocities, and electric/magnetic fields. In engineering, for modeling physical systems, such as the analysis of electrical circuits or mechanical structures.

**2. Computer Graphics and Computer Vision:** In computer graphics, vectors are used to represent positions, directions, and transformations in 2D and 3D spaces. In computer vision, vectors represent features extracted from images, aiding in tasks like object recognition and tracking.

**3. Quantum Mechanics:** Quantum mechanics extensively uses vector spaces to model states of particles and systems. The wave functions that describe these states often exist in complex vector spaces, allowing for calculations of probabilities and behavior of particles.

**4. Economics and Finance:** Vector spaces are employed in economics to model preferences, resources, and allocations. In finance, they're used to represent portfolios, asset returns, and risk analysis, aiding in portfolio optimization and risk management.

**5. Machine Learning and Data Analysis:** Vector spaces play a crucial role in machine learning and data analysis. Vectors represent features in datasets, enabling algorithms to learn patterns, classify data, and make predictions. Techniques like support vector machines and neural networks operate in vector spaces.

**6. Control Systems and Robotics:** In control systems, vectors help describe states, inputs, and outputs of dynamic systems. In robotics, they're used to represent positions, orientations, and movements of robots, aiding in path planning and control algorithms.

**7. Signal Processing and Communication:** Vectors are used to represent signals in various domains (time, frequency), facilitating analysis and processing of audio, images, and other types of signals. In communication systems, vector spaces model signal spaces, encoding, and decoding schemes.

These applications demonstrate the versatility and significance of vector spaces across diverse disciplines, providing a powerful framework for understanding and solving complex problems.

## BASIC CONCEPTS AND DEFINITIONS:

### SCALAR:

It is an element of a Field which is used to define a vector space

It is a physical quantity that can be described by a single element of number field such as “Real number”.

Therefore a scalar is a Real number

Scalars form the foundation of many programming concepts and are essential for building algorithms, manipulating data, and controlling program flow.

**FIELDS:**

A Field F is a non-empty set of elements equipped with two operations: addition(+) and multiplication(.) satisfying the following properties for all elements $a, b$ and $c$ in the field:

1. Closure: For any $a, b \in F, a + b \in F$ $and$ $a \cdot b \in F$

2. Commutativity: Addition and multiplication are commutative I.e., $a + b = b + a$ $and$ $a \cdot b = b \cdot a$ for all $a$ and $b$ in the field.

3. Associativity: Addition and multiplication are associatiove, meaning $(a + b) + c = a + (b + c) and$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

4. Additive Identity:

There exists an element $0 \in F$such that $a + 0 = a$

5. Multiplicative Identity: There exists an element $1 \in F$ such that $a \cdot 1 = a$

6. Additive Inverses: For every $a \in F,$ there exists an element $-a \in F$ such that $a + (-a) = 0.$

7. Multiplicative Inverses: For every $a \neq 0 \in F,$ there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1.$

Common examples of fields include

- The set of real numbers (R)
- The set of complex numbers(C), and
- The set of rational numbers (Q).

  These ste satisfy all the properties mentioned above, amking them fields

Set of real numbers( R) is most commonly used field

The set of Integers (I)

Is not a Field

Because Non-Closure under Multiplicative inverses:

Closure under multiplicative inverses means for any non-zero elemnt $a \in I$ its inverse $\frac{1}{a} \in I$

In $I$, if you take the inverse of integers other than 1, the result ( if it exists at all) is not an integer.

## VECTOR:

**Vector** is a quantity which has magnitude and direction

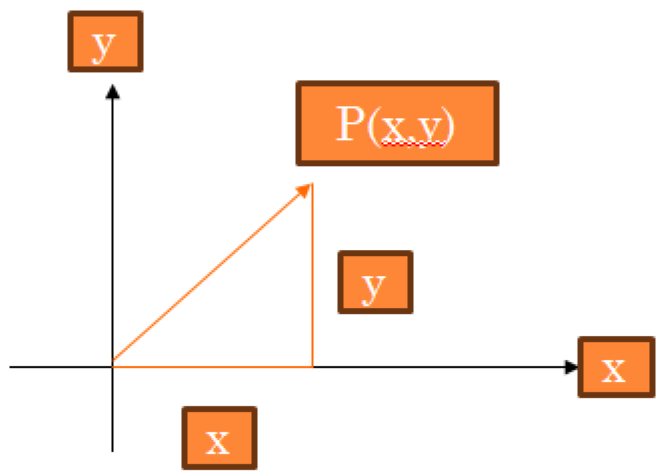
Vector = Magnitude + Direction

If a car is moving 60 Km/hr in the east direction

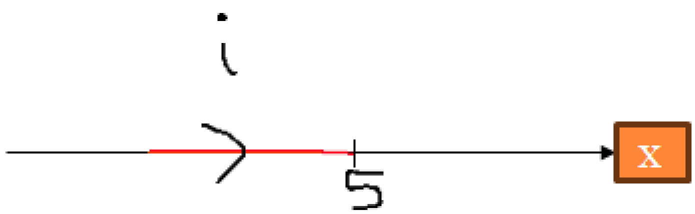
Then 60 is magnitude( scalar)

East is the direction

**2- DIMENSIONVECTOR:** Let $P(x, y)$ be a point in the $R^2$- plane. Then the position vector of the point $P = (x, y)$ is $\overrightarrow{OP}$ and is referred as column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.



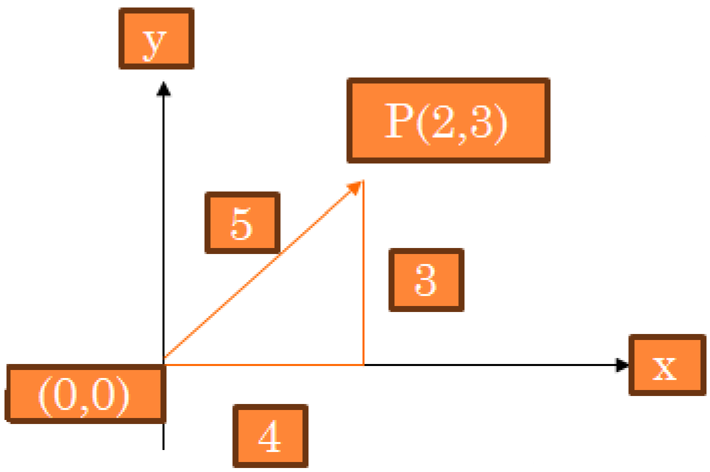

Let $\vec{v} = 5i$, I is unit vector along $x - axis$
5 represents the length of the vector
$i$ represents the direction

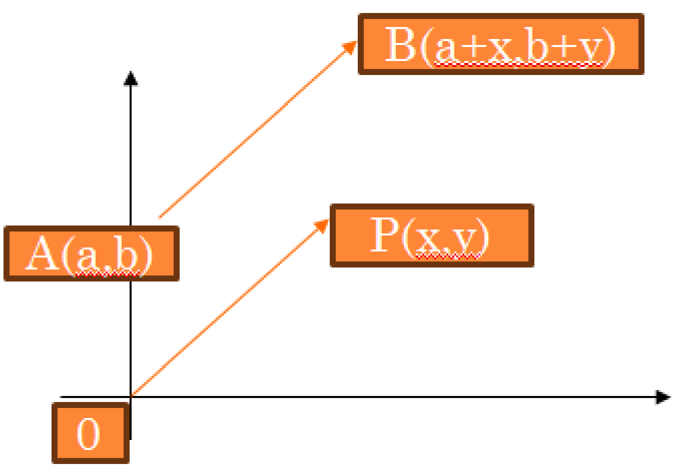



2- Dimension Vector:
Length of $\vec{v} = 4i + 3j$
$|\vec{v}| = \sqrt{2^2 + 3^2} = 5 = v(scalar)$
In general:
Length of $\vec{v} = xi + yj$

$|\vec{v}| = \sqrt{x^2 + y^2} = v$ this is nothing but

**Pythagoras Theorem**



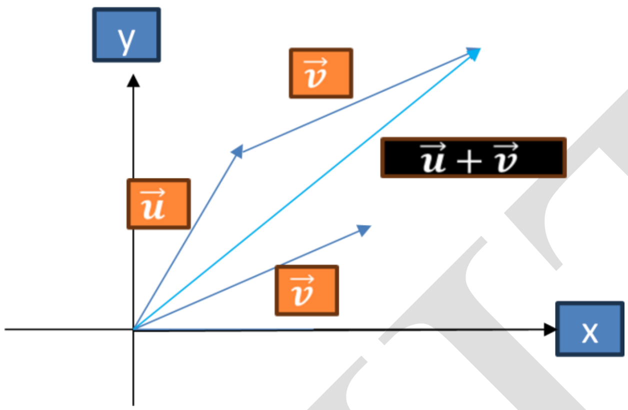




Is $\overrightarrow{AB} = \overrightarrow{OP}$
Yes $\overrightarrow{AB} = \overrightarrow{OP}$

If the length (magnitude) of a vector $\vec{u}$ is the same as the length of a vector $\vec{v}$, and they point in the same direction in space , then $\vec{u}$ and $\vec{v}$ are equal vectors.

## ADDITION OF VECTORS:

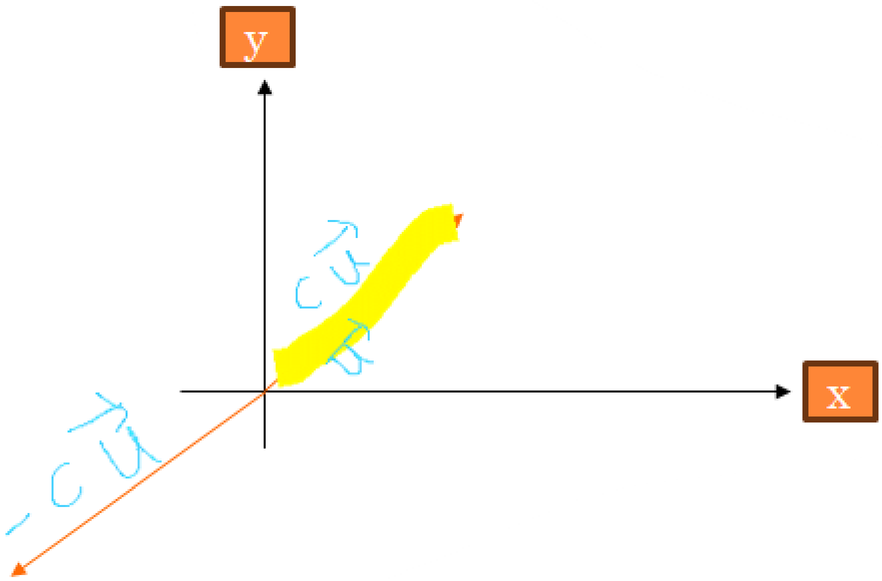
Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$




## SCALAR MULTIPLICATION OF A VECTOR

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $C \in R$ be any scalar then $C\vec{u} = \begin{bmatrix} Cu_1 \\ Cu_2 \end{bmatrix}$

$C\vec{u}$ magnifies the length in the same direction. $-C\vec{u}$ magnifies the length in the opposite direction




## VECTOR SPACE:

Let non-empty collection of vectors (in $R^n$, where $n \in \{1,2,3,\ldots\}$ along with two operations addition (+) and scalar multiplication ($\cdot$) defined over a field F is called VECTOR SPACE ($V + \cdot$), if it satisfies the following axioms

i. Closure under Addition: For all $u, v \in V, u + v \in V$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ where $u_i$ *and* $v_i \in F$, what is a field

(Field may be R/Q/C) then $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

ii. Closure under scalar multiplication:

For all $v \in V$ and $c \in F, cv \in V$

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$,then $c.u = \begin{bmatrix} c.u_1 \\ c.u_2 \\ \vdots \\ c.u_n \end{bmatrix} \in V$

iii. Additive Identity: There exists a $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in V$

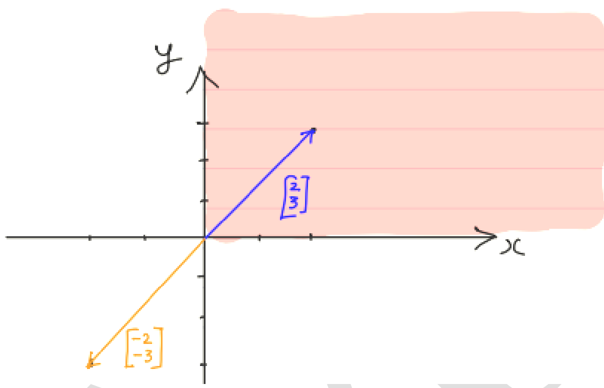
If the set of vectors $(V+.)$ satisfies the above axioms then it is called vector space over a field F.

**Examples:**

1. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \right\}$ a vector space with usual addition and scalar multiplication?

Ans: No

Since if we consider $\propto = -1$, then $\propto . \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin V$.

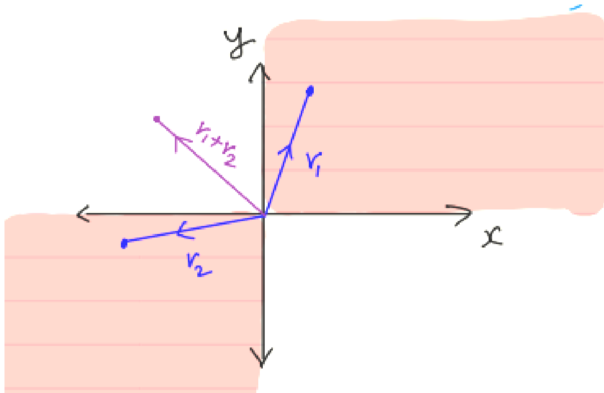


2. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \; or \; x \leq 0, y \leq 0 \right\}$ a vector space?

Ans: No

In fig $v_1, v_2 \in V$ not $v_1 + v_2$

Let $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, v_1 + v_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \notin V$




3. Is $V = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \; or \; x \leq 0, y \leq 0 \; or \; x \leq 0 \; and \; y \geq 0 \right\}$ a vector space?

4. Let $M_2 = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in R \right\}$ is a vector space with usual matrix addition and scalar multiplication defined by

i. $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

ii. $\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$ for every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$ and $\alpha \in R$

5. In general $R^{m \times n}$ set of real $m \times n$ matrices is a vector space under usual matrix addition and scalar multiplication.

Set of all polynomial of degree $\leq 2$

$P_2 = \{a + bx + cx^2 | a, b, c \in R\}$

With binary operations + and · defined by

i. $(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2$

ii. $\alpha(a_1 + b_1x + c_1x^2) = \alpha a_1 + (\alpha b_1)x + (\alpha c_1)x^2$

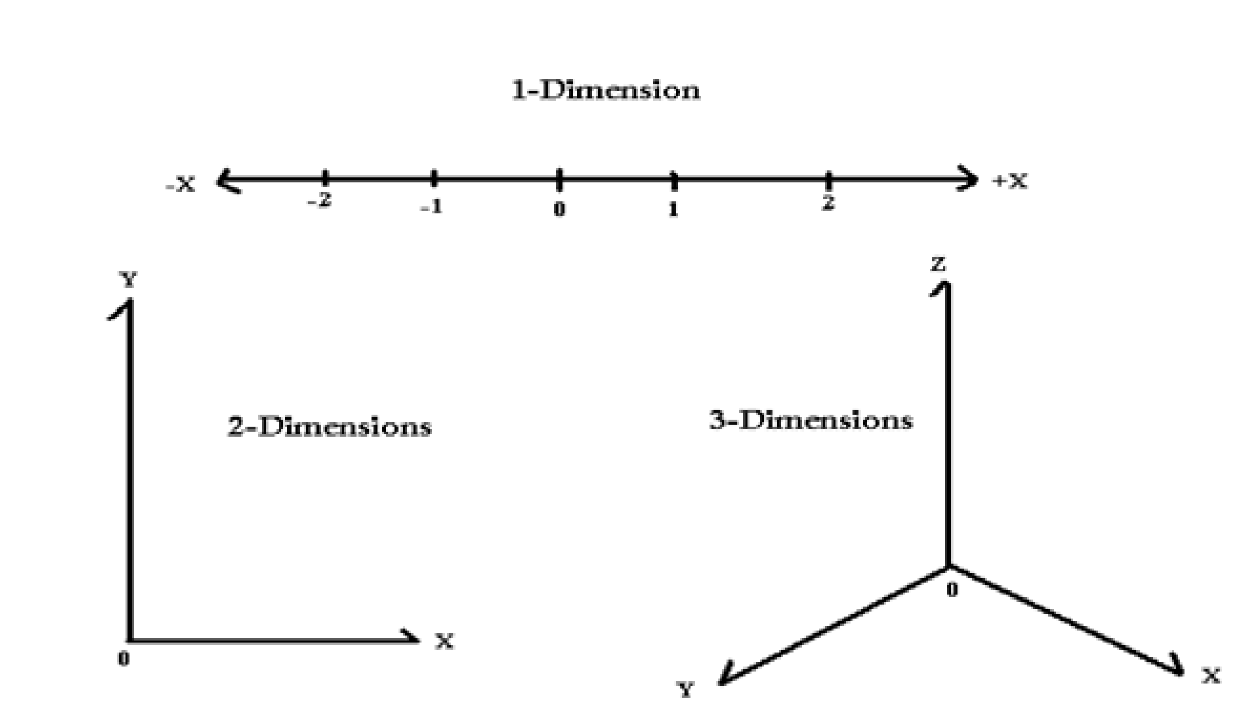
for every $(a_i + b_ix + c_ix^2) \in P_2$ and $\alpha \in R$ where i=1,2,…

Hence zero vector is $0 + 0x + 0x^2$

6. Is this a vector space

i. Let $M = \left\{\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \middle| a, b, c, \in R \right\}$ is a vector space with usual matrix addition and scalar multiplication

ii. Let $M = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \right\}$ is a vector space with usual matrix addition and scalar multiplication.




**VECTOR SUBSPACE:** A subset W of a vector space V over the real numbers $R$ is called a subspace of $V$ if $W$ is a vector space over $R$ with the operations of addition and scalar multiplication defined on V.

Let V is a vector space over a field F and a W is a non-empty subset of V. Then, W is said to be subspace of vector space V if:

- $u, v \in W$ then $u + v \in W$
- $u \in W$ and $k \in F$ then $ku \in W$

Then W is said to be subspace of V

**EXAMPLES:**

1. Lets obtain all subspace of $R^2 = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in R\right.$ under usual binary operations. Subspace of $R^2$.

a) All of $R^2$ (Improper Subspace)

b) $S = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, Zero vector only. (Trivial Subspace)

c) Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. Find all subspace of $R^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x, y, z \in R \right.$. Under usual binary operations .

a) All of $R^3$ (Improper Subspace)

b) Zero vector only; that is $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

c) All lines through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d) All planes through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

## Remarks:

i. Every subspace of a vector space is a vector space.

ii. Any vector space $V$ automatically contains two subspaces

a) The set $\{0\}$, set consisting of only zero vector, is called trivial subspace.

b) $V$ itself is called improper subspace.

## PROBLEMS:

**1. Given $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} | x_1 = x_2 \text{ and } x_i \in R \right\}$. Verify S is a subspace of $R^3$.**

**Solution:**

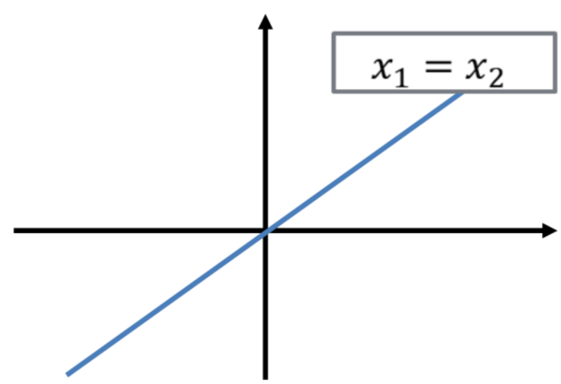
i. **Closure under Vector Addition:**

let $u, v \in S$

$u = \begin{bmatrix} a \\ a \\ b \end{bmatrix}, v = \begin{bmatrix} c \\ c \\ d \end{bmatrix}$

i. $u + v = \begin{bmatrix} a + c \\ a + c \\ b + d \end{bmatrix} \in S$

ii. $\alpha. \begin{bmatrix} a \\ a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha a \\ \alpha b \end{bmatrix} \in S$, for any $\alpha \in \mathbb{R}$



iii. $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$

$S$ is a subspace of $\mathbb{R}^3$.

**2. Consider a set of vectors in three-dimensional space, $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \right\}$. Determine whether this set forms a vector space. If it does, prove it; if not, provide a counterexample for one of the vector space axioms.**

**Solution:**

To determine whether the set of vectors V = {(1, 2, 3), (0, -1, 2), (-2, 1, 4)} forms a vector space, we need to check whether it satisfies the vector space axioms.

**Closure under Vector Addition:**

To check this, we need to ensure that the sum of any two vectors in V is also in V.

Let's take two arbitrary vectors from V:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

u + v = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

The result $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ is not in the set V, so closure under vector addition is violated.

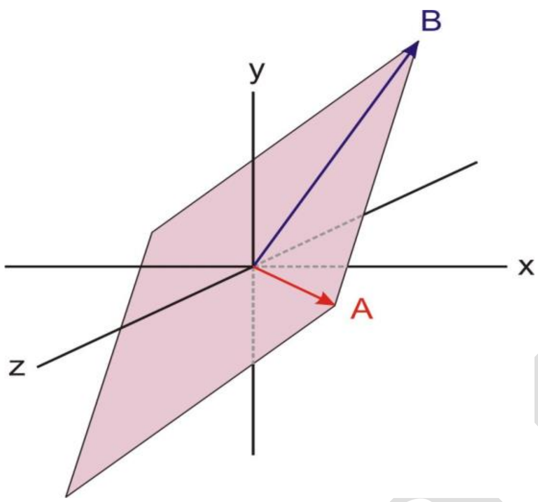
Therefore, V does not form a vector space because it doesn't satisfy the closure property under vector addition, which is one of the fundamental axioms of vector spaces.

However, there is no vector in V such that when added to any other vector in V, it results in the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This violates the existence of an additive identity.

As we found counterexamples for the closure propertie and the existence of an additive identity, we can conclude that the set V does not form a vector space.

**3. Consider the set of vectors S = {(x, y, z) | x, y, z ∈ ℝ, 2x + 3y - z = 0}. Determine whether the set S defined above is a vector space. Specifically, check if it satisfies all the conditions If it is a vector space, prove it. If it is not, provide a counterexample for one of the conditions.**

**Solution:**




a) Closure under Vector Addition: Consider two vectors $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $and$ $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in S.

Their sum is $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$. To check closure, we need to verify if $2(x_1 + x_2) + 3(y_1 + y_2) - (z_1 + z_2) = 0$.

Let's calculate: $2(x_1 + x_2) + 3(y_1 + y_2) - (z_1 + z_2) = 2x_1 + 2x_2 + 3y_1 + 3y_2 - z_1 - z_2$

Since $2x_1 + 3y_1 - z_1 = 0$ (because $(x_1, y_1, z_1)$ is in S) and $2x_2 + 3y_2 - z_2 = 0$ (because $(x_2, y_2, z_2)$ is in S), the expression simplifies to: 0 + 0 - 0 = 0

So, S is closed under vector addition.

b) Closure under Scalar Multiplication: Consider a vector $(x_1, y_1, z_1)$ in S and a scalar α. The product $\alpha \cdot (x_1, y_1, z_1)\ is\ (\alpha x_1, \alpha y_1, \alpha z_1)$. To check closure, we need to verify if $2(\alpha x_1) + 3(\alpha y_1) - (\alpha z_1) = 0$.

Let's calculate:

$$2(\alpha x_1) + 3(\alpha y_1) - (\alpha z_1) = \alpha(2x_1 + 3y_1 - z_1)$$

Since $2x_1 + 3y_1 - z_1 = 0$ (because $(x_1, y_1, z_1)$ is in S), the expression simplifies to:

$$\alpha(0) = 0$$

So, S is closed under scalar multiplication.

c) Identity Element of Vector Addition: The zero vector in $\mathbb{R}^3$, which is (0, 0, 0), is also in S. This property holds.

**4. Show that the subset $W = \{(x_1, x_2, x_3)^T | x_1 + x_2 + x_3 = 0\}$ of the vector space $V_3(R)$ is a subspace of $V_3(R)$.**

**Solution:** Let

$\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3)$ be any two elements of W.

$\therefore\ x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$ … … … (1)

Consider,

$$c_1.\alpha + c_2.\beta = c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3)$$
$$= (c_1x_1, c_1x_2, c_1x_3) + (c_2y_1, c_2y_2, c_2y_3)$$
$$= (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3)$$

To show that $c_1.\alpha + c_2.\beta \in W$, we have to show that the sum of the components of $c_1.\alpha + c_2.\beta$ is zero.

Consider

$$c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3 = c_1(x_1 + x_2 + x_3) + c_2(y_1 + y_2 + y_3)$$
$$= c_1.0 + c_2.0 = 0$$

$\therefore\ c_1.\alpha + c_2.\beta \in W$, and hence W is a subspace of $V_3(R)$.

**EXCERSIS:**

**1. Determine whether or not the $W = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 z = 0\}$ subset of $R^3$ are subspace.**

**2. Determine whether or not the $W = \{(a, b, c, d) | a + b = 1\}$ subset of $R^4$ are subspace.**

**3. Given $S = \left\{\begin{bmatrix} x \\ 1 \end{bmatrix} \middle| x \in R\right\}$. Check whether S is a subspace of $R^2$. Represent the given set geometrically.**

## Remarks:

1. If $W_1$ & $W_2$ are two subspaces of a vector space $V$ then $W_1 \cap W_2$ is also a subspace of V.

2. If $W_1$ & $W_2$ are two subspaces of a vector space $V$ then $W_1 \cup W_2$ is a subspace of V iff $W_1 \subset W_2$ or $W_2 \subset W_1$.

## LINEAR COMBINATION:

**Definition: Let V be a vector space over the field F and $v_1, v_2, \dots, v_n \in V$ then a vector 'u' of the form, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = u$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$,(some of them may be zero)then 'u' is called a linear combination of the vectors $v_1, v_2, \dots, v_n$.**

**Example:**

Let $(1,2,1)^T, (-1,0,2)^T$ and $(-1,4,8)^T$ be three vectors in $R^3$.

$2(1,2,1)^T + 3(-1,0,2)^T = (-1,4,8)^T$

$(-1,4,8)^T$ is a linear combination of the vectors $(1,2,1)^T, (-1,0,2)^T$.

## PROBLEMS:

**1. Is $[1, -3, 4]^T$ is a linear combination of $[0, -1, 5]^T$ and $[-2, 3, 4]^T$ .**

**Solution:**

If $[1,-3,4]^T = \alpha_1 [0,-1,5]^T + \alpha_2 [-2,3,4]^T$ holds

To find $\alpha_1$ and $\alpha_2$ such that the above condition holds

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$0\alpha_1 - 2\alpha_2 = 1$$

$$-1\alpha_1 + 3\alpha_2 = -3$$

$$5\alpha_1 + 4\alpha_2 = 4$$

This represents a system of linear equations of the form $A\alpha = B$

Where $A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 5 & 4 \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

On solving the above system of equations by matrix method, we get

$$\begin{bmatrix} -1 & 3: & -3 \\ 0 & -2: & 1 \\ 0 & 0: & -3 \end{bmatrix}$$

$$\rho(A) \neq \rho(A:B)$$

There is no such $\alpha_1$and $\alpha_2$ such that

There it is not a linear combination.

**2. Express the vectors $(3,5,2)^T$ as a linear combination of the vectors $(1,1,0)^T, (2,3,0)^T, (0,0,1)^T$ of $V_3(R)$.**

**Solution:** Let

$(3,5,2) = \alpha_1(1,1,0) + \alpha_2(2,3,0) + \alpha_3(0,0,1)$

$= (\alpha_1, \alpha_1, 0) + (\alpha_2 2, \alpha_2 3, 0) + (0,0,\alpha_3)$

$= (\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, \alpha_3)$

$\Rightarrow \alpha_1 + 2\alpha_2 = 3;\ \alpha_1 + 3\alpha_2 = 5; \alpha_3 = 2$

Solving this system, for $\alpha_1, \alpha_2, \alpha_3$, we get $\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = 2$

$\therefore (3,5,2) = -1(1,1,0) + 2(2,3,0) + 2(0,0,1)$.

**3. Write the vector $v = (4,2,1)$ as a linear combination of the vectors $u_1 = (1,-3,1), u_2 = (0,1,2), u_3 = (5,1,57)$**

**Solution:** Let $v = c_1u_1 + c_2u_2 + c_3u_3$

$(4,2,1) = c_1(1,-3,1) + c_2(0,1,2) + c_3(5,1,57)$

$c_1 + 0c_2 + 5c_3 = 4$
$-3c_1 + c_2 + c_3 = 2$

$c_1 + 2c_2 + 37c_3 = 1$

Let $AX = B$ where $A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 37 \end{bmatrix}$, X=$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Consider, $[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 37 & 1 \end{array}\right]$ $R_2 \to R_2 + 3R_1, R_3 \to R_3 - R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & 32 & -3 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & 0 & -31 \end{array}\right]$$

$$Rank[A] = 2 \neq Rank[A:B] = 3$$

Hence the system of linear equations is consistent. i.e, solution doesn't exist.

Therefore, $v$ cannot be expressed as linear combination of the vectors $u_1, u_2, u_3$.

**EXCERSISE:**

1. **Express the vectors $(2, -1, -8)$ as a linear combination of the vectors $(1, 2, 1), (1, 1, -1), (4, 5, -2)$ of $V_3(R)$.**

   **ANS: $(2, -1, -8) = -3(1, 2, 1) + 5(1, 1, -1) + 0(4, 5, -2)$**

2. **Express the vectors $(1, -2, -5)$ as a linear combination of the vectors $(1, 1, 1), (1, 2, 3), (2, -1, 1)$ of $V_3(R)$.**

   **ANS: $(1, -2, -5) = -6(1, 1, 1) + 3(1, 2, 3) - (2, -1, 1)$**

3. **For which value of k will the vector $(1, -2, k)$ in $R^3$ be a linear combination of the vectors $(3, 0, -2)$ and $(2, -1, -5)$.**

   **ANS: k=-8**

4. **Let V be a vector space, and consider the vectors u = (2, 1), v = (0, -1), and w = (-1, 3) in V. Determine whether the vector x = (1, 2) is a linear combination of u, v, and w. If it is, provide the coefficients that make it a linear combination. If it is not, explain why.**

## LINEAR INDEPENDENT AND DEPENDENT:

A set of vectors $\{v_1, v_2, ..., v_n\}$ is **linearly independent** if the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ has only the trivial solution, where $c_1 = c_2 = \cdots = c_n = 0$.

A set of vectors $\{v_1, v_2... v_n\}$ is **linearly dependent** if there exist coefficients $c_1, c_2... c_n$, not all zero, such that the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ has a non-trivial solution, where at least one coefficient is non-zero.

### PROBLEMS:

**1. Examine the vectors $\boldsymbol{u} = (1, 2, -1)^T, \boldsymbol{v} = (-2, 0, 1)^T$ and $\boldsymbol{w} = (3, -1, 2)^T$ belonging to the vector space $V_3(R)$ for linear dependency.**

**Solution:**

To find a linear combination of vectors $u, v$ and $w$ that result in the zero vectors, we need to find coefficient $c_1, c_2$ and $c_3$ such that:

$c_1u + c_2v + c_3w = (0,0,0)^T$

We have the following vectors:

$u = (1,2,-1)^T, v = (-2,0,1)^T$ and $w = (3,-1,2)^T$

Now, let's set up the equations based on the linear combination:

$c_1u + c_2v + c_3w = c_1(1,2,-1)^T + c_2(-2,0,1)^T + c_3(3,-1,2)^T = (0,0,0)^T$

We have a system of linear equations:

$c_1 - 2c_2 + 3c_3=0$

$2c_1 - c_3=0$

$-c_1 + c_2 + 2c_3=0$

On solving,

$c_1 = 0, c_2 = 0$ and $c_3 = 0$

In other words, the coefficients are all zero, and the linear combination is:

$0u + 0v + 0w=(0,0,0)$

Linearly independent

**2. Examine whether the vectors $\boldsymbol{u} = (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{0})^{\boldsymbol{T}}, \boldsymbol{v} = (\mathbf{2}, \mathbf{4}, \mathbf{2}, \mathbf{0})^{\boldsymbol{T}}$, and $\boldsymbol{w} = (\mathbf{3}, \mathbf{6}, \mathbf{3}, \mathbf{0})^{\boldsymbol{T}}$ are linearly independent or linearly dependent in $\mathbb{R}^4$ . Provide an explanation and any necessary calculations.**

**Solution:**

To determine whether the vectors $u = (1, 2, 1, 0)^T, v = (2, 4, 2, 0)^T$, and $w = (3, 6, 3, 0)^T$ are linearly independent or linearly dependent in $\mathbb{R}^4$ , we need to check if there exist constants $a, b$, and $c$, not all equal to zero, such that:

$$au + bv + cw = (0, 0, 0, 0)$$

Let's set up and solve a system of equations to find these constants:

$$a(1, 2, 1, 0) + b(2, 4, 2, 0) + c(3, 6, 3, 0) = (0, 0, 0, 0)$$

This leads to the following system of equations:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$
$$a + 2b + 3c = 0$$
$$0 = 0$$

Notice that equations 1 and 3 are identical, which means there are only three unique equations in the system. Now, we can simplify the equations:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$
$$0 = 0$$

Let's simplify equations 1 and 2 further:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$

Divide equation 2 by 2:

$$2a/2 + 4b/2 + 6c/2 = 0$$
$$a + 2b + 3c = 0$$

Now, we see that equations 1 and 2 are equivalent, so we have two identical equations:

$$a + 2b + 3c = 0$$
$$a + 2b + 3c = 0$$

Since we have only two distinct equations for three variables ($a, b$, and $c$), there are infinitely many solutions that satisfy this system. This indicates that the vectors $u, v$, and $w$ are linearly dependent in $\mathbb{R}^4$ because we can find non-trivial constants $a, b$, and $c$ that make the linear combination equal to the zero vector.

**3. Determine whether the vectors $(\mathbf{1}, \mathbf{2}, \mathbf{3})^T, (\mathbf{3}, \mathbf{1}, \mathbf{7})^T, (\mathbf{2}, \mathbf{5}, \mathbf{8})^T$ are linearly dependent or linearly independent.**

**Solution:** Let $c_1u_1 + c_2u_2 + c_3u_3 = 0$

$$c_1(1,2,3) + c_2(3,1,7) + c_3(2,5,8) = 0$$

$$c_1 + 3c_2 + 2c_3 = 0$$
$$2c_1 + c_2 + 5c_3 = 0$$
$$3c_1 + 7c_2 + 8c_3 = 0$$

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$, $X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Consider, $[A] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix} R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$[A] = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix} R_2 \to -R_2, R_3 \to -R_3/2$$

$$[A] = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_2 \to R_3$$

$$[A] = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix} R_3 \to R_3 - 5R_2$$

$$[A] = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$Rank[A] = 3 = number\ of\ unknowns$$

Unique solution exists.

$$i.e, c_1 = 0, c_2 = 0, c_3 = 0$$

So vectors are linearly independent.

**4. Determine whether the vectors $(1,4,9)^T, (3,1,4)^T, (9,3,12)^T$ are linearly dependent or linearly independent.**

**Solution:** Let $c_1u_1 + c_2u_2 + c_3u_3 = 0$

$c_1(1,4,9) + c_2(3,1,4) + c_3(9,3,12) = 0$

$c_1 + 3c_2 + 9c_3 = 0$
$4c_1 + c_2 + 3c_3 = 0$
$9c_1 + 4c_2 + 12c_3 = 0$

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$, $X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Consider, $[A] = \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix} R_2 \to -R_2/11, R_3 \to -R_3/29$

$$[A] = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$[A] = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Rank[A] = 2 < 3 = number\ of\ unknowns$$

One unknown is constant. So the vectors are linearly dependent.

**5. Verify that the vectors $v_1 = (1,1,1), v_2 = (1,0,1), v_3 = (0,1,1)$ are linearly independent.**

Solution: Consider $x_1v_1 + x_2v_2 + x_3v_3 = 0$

$$\Rightarrow x_1(1,1,1) + x_2(1,0,1) + x_3(0,1,1) = 0$$

$$\Rightarrow (x_1, x_1, x_1) + (x_2, 0, x_2) + (0, x_3, x_3) = 0$$

$$\Rightarrow (x_1 + x_2, x_1 + x_3, x_1 + x_2 + x_3) = (0,0,0)$$

$$\Rightarrow x_1 + x_2 = 0 \rightarrow (i)$$

$$x_1 + x_3 = 0 \longrightarrow (ii)$$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (iii)$$

By solving (i) and (iii) we obtain $x_3 = 0 \Rightarrow x_1 = 0, x_2 = 0$

Since $x_1 = x_2 = x_3 = 0$, the vectors are Linearly independent.

**EXCERSISE:**

1. **Verify that the set $S = \{(1,0,1), (1,1,0), (-1,0,-1)\}$ is linearly dependent in $V_3(R)$.**
2. **Verify that the vectors $(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0)$ and $(2,1,1,6)$ are linearly dependent in $R^4$.**

## Remarks:

**Linearly dependent vectors lies on the same line in $\mathbb{R}^2$ and same plane in $\mathbb{R}^3$.**

## LINEAR SPAN OF A SET:

**Definition: Let V be a vector space and $v_1, v_2, \ldots, v_n$ be vectors in V. The set of all linear combinations of $v_1, v_2, \ldots, v_n$ is called span of $v_1, v_2, \ldots, v_n$.It is denoted by**

**Span**$(v_1, v_2, \dots, v_n)$ **or L**$(v_1, v_2, \dots, v_n)$**. In other words** $Span(V) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n |\ \alpha_i \in R, i = 1, 2, 3, \dots, n$.

**Example:**

$$\text{Span}\{(1,-1,4)^T, (2,-1,0)^T\} = \left\{\propto_1 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \propto_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \middle| \forall \propto_1, \propto_2 \in \mathbb{R}\right\}$$

Linear combination is $3\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} \in \text{Span}\{(1,-1,4)^T, (2,-1,0)^T\}$

**Example:** Find linear span of $e_1$ and $e_2$ in $\mathbb{R}^3$

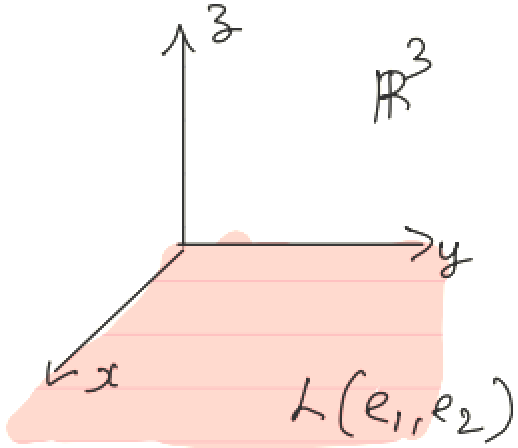
Solution: $L(e_1, e_2) = \{\propto_1 e_1 + \propto_2 e_2 | \forall \propto_1, \propto_2 \in \mathbb{R}\}$

$$\left\{\propto_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \propto_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} | \forall \propto_1, \propto_2 \in \mathbb{R}\right\}$$

$$\left\{\begin{bmatrix} \propto_1 \\ \propto_2 \\ 0 \end{bmatrix} | \forall \propto_1, \propto_2 \in \mathbb{R}\right\}$$

Is set of all points in the xy-plane

It is a subspace of $\mathbb{R}^3$




## LINEAR SPANNING SET FOR VECTOR SPACE:

Definition: The set $S = \{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ is called spanning set for vector space V if and only if $span\,(S) = V$

In other words, if every vector in V can be written as linear combination of $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$, then $S$ is called a spanning set of $V$.

Example: 1. The set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ span $\mathbb{R}^3$.

Solution: $\propto_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \propto_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + \propto_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\propto_1\\\propto_2\\\propto_3\end{bmatrix}$

Thus $S$ is spanning set of $\mathbb{R}^3$.

## BASIS AND DIMENSION

**Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis for a vector space V iff**
**$i$) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent**
**$ii$)Span $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \mathbf{V}$**

Example:

1. Let $\mathrm{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathrm{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathrm{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ then $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ form a basis for $\mathbb{R}^3$

What about $\{\mathrm{e}_1, \mathrm{e}_2\}$ is this forms a basis for $\mathbb{R}^3$

2. Even $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$ is basis for $\mathbb{R}^3$.

In fact, there are infinitly many bases for a vector space

**NOTE:**

1. In $\mathbb{R}^2$ there must have 2 vectors.
2. In $\mathbb{R}^3$ there must have 3 vectors
2. Vectors in a 3D space have three components. They can be represented as $(x, y, z)$, where $x, y,$and $z$ are the components along the three coordinate axes. The vector space is denoted as $\mathbb{R}^3$.

In general, for an $n$-dimensional vector space, vectors have $n$ components and can be represented as $(x_1, x_2, \ldots, x_n)$. The dimension of the vector space in $n$, and it is denoted as $\mathbb{R}^n$ if the vectors consist of real numbers.

## PROBLEM:

**1. Show that the vectors $(1,1,2,4)^T, (2,-1,-5,2)^T, (1,-1,-4,0)^T$ and $(2,1,1,6)^T$ are linearly dependent in $R^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.**

**Solution:**

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 4R_1$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} R_2 \to -R_2, R_2 \to -\frac{1}{3}R_2, R_2 \to -\frac{1}{2}R_2$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$$

The final matrix is the required echelon form. The rank of A =2. Thus the given vectors are linearly dependent.

Corresponding to non-zero rows in the final matrix the rows of $A$ are$(1,1,2,4), (2,-1,-5,2)$.

These two form a set of linearly independent subset. The dimension of the subspace spanned by these vectors is 2.

Further these two vectors form a basis of the subspace.

2. **Find the Dimension and basis for Matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$**

Solution: W. K. T, Dimension = Rank (A)

∴ It is necessary to reduced to echelon form of given matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_1$$

Dimension (A) = Rank (A) = 2

Basis of Matrix (A) = Number of columns with pivot elements

∴ Basis of Matrix $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

## Exercise

1. Show that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

2. Consider the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Determine if these vectors form a basis for $\mathbb{R}^3$

3. Find the dimension and basis of the subspace spanned by the vectors

   $(2,4,2), (1,-1,0), (1,2,1)$ and $(0,3,1)$ in $R_3$.

   ANS: Basis= $\{(1,2,1), (0,3,1)\}$ Dimension=2.

4. Find the basis and dimension of the subspace of V spanned by the subset S is given by

i) S= $\{(2,1), (1,-1), (0,2)\}$, $V = R^2$

ANS: Basis=$\{(2,1), (1,-1)\}$ Dimension=2.

ii) $S = \{(1,1,1), (1,2,3), (-1,0,1)\}$, $V = R^3$

ANS: Basis=$\{(1,1,1),(1,2,3)\}$ Dimension=2.

iii) $S = \{(1,-2,3,-1),(1,1,-2,3\}, \ V = R^4$

ANS: Basis=$S$ , Dimension=2.

## Application problem:

**1. Given two emails with feature vectors (6 words, 1 image, 4 links) and (3 words, 2 images, 1 link), weights assigned to these features is given by (0.5, 0.25, 0.25). Assume the emails are labeled as NOT SPAM (0) and SPAM (1).**

**a) Name the mathematical concept that is involved in weighted sum**

**b) Evaluate the weighted sum and classify the email as SPAM or NOTSPAM**

**c) Identify two practical applications of Linear Algebra**

**Solution** :

a) The mathematical concept involved in the weighted sum is a linear combination. In this context, the weighted sum can be expressed as a linear combination of the feature vectors, where each feature is multiplied by its corresponding weight.

b) To evaluate the weighted sum for classifying the email as SPAM or NOT SPAM:

Let's denote the feature vector for an email as $X = (x_1, x_2, x_3)$ where $x_1$ is the number of words , $x_2$ is the number of images, and $x_3$ is the number of links.

Given the weights $(w_1, w_2, w_3)$ for word count, image count, and link count, respectively,

The vector representation of the weighted sum for an email can be calculated as follows:

Weighted sum= $x_1.w_1 + x_2.w_2 + x_3.w_3$

The given weights are $(0.5,0.25,0.25)$

Email 1: (6 words, 1 image, 4 links)

Email 2: (3 words, 2 images, 1 link)

The weighted sum can be calculated as follows:

**Email1:**

Weighted sum = (6 words * 0.5) + (1 image * 0.25) + (4 links * 0.25)

Weighted Sum=3+0.25+1

Weighted Sum=4.25

**Email 2:**

Weighted sum = (3 words * 0.5) + (2 image * 0.25) + (1 links * 0.25)

Weighted Sum=1,5+0.5+0.25

Weighted Sum=2.25

If probability (algorithm(weighted sum))$\geq$ 0.5 SPAM [ because we have labeled SPAM(1)]

If probability (algorithm(weighted sum))$<$ 0.5 NOTSPAM [ because we have labeled NOTSPAM(0)]

Note: Algorithms will be taught in ADVANCED LINEAR ALGEBRA

c) Two practical applications of linear algebra (as mentioned earlier):

1. Graphics and image processing
2. Machine learning and artificial intelligence
3. Cryptography: Linear algebra is used in various cryptographic algorithms. For example, techniques such as the RSA algorithm rely on the properties of matrices and modular arithmetic for secure data encryption and decryption.
4. Medical Imaging: Linear algebra is widely applied in medical imaging techniques like MRI and CT scans. Transformations and reconstructions of images are often formulated as linear algebra problems. Techniques such as image registration and homographic reconstruction involve the use of matrices and vectors for processing medical images.

**2. Sentiment Analysis of Movie Reviews:**

**A dataset containing movie reviews (50 positive, 30 negative, 10 neutral sentiment) along with their corresponding sentiments (positive/ negative/ neutral sentiment) and weights assigned to these sentiments is given by (0.5, 0.5, 0). Assume the reviews are labeled as POSITIVE (1) and NEGATIVE (0).**

**a. Name the mathematical concept that is involved in weighted sum**

**b. Evaluate the weighted sum and classify the review as POSITIVE and NEGATIVE**

**c. Identify two practical applications of Linear Algebra**

**Solution:**

a) The mathematical concept involved in weighted sum is linear combination . In linear algebra, a linear combination of variables is the sum of these variables each multiplied by a weight or coefficient.

b) To evaluate the weighted sum and classify the review as POSITIVE or NEGATIVE

Let's denote the feature vector for reviews as $X = (x_1, x_2, x_3)$ where $x_1$ is the positive reviews , $x_2$ is the negative reviews, and $x_3$ is the neutral sentiments.

Given the weights $(w_1, w_2, w_3)$ for positive, negative, and neutral sentiments respectively,

The vector representation of the weighted sum for reviews can be calculated as follows:

Weighted sum= $x_1.w_1 + x_2.w_2 + x_3.w_3$

The given weights are (0.5, 0.5, 0)

Weighted sum=(positive*0.5)+(negative*0.25)+(neutral*0)

Weighted sum=50*0.5+30*0.5+0

Weighted sum=40

If probability (algorithm (weighted sum))$\geq$ 0.5 POSITIVE[ because we have labeled POSITIVE(1)]

If probability (algorithm (weighted sum))$<$ 0.5 NEGATIVE [ because we have labeled NEGATIVE(0)]

NOTE: Algorithms will be taught in ADVANCED LINEAR ALGEBRA

c) Two practical applications:

**Computer Graphics**: Linear algebra is widely used in computer graphics to perform operations such as rotation, translation, scaling, and projection. Matrices and vectors are fundamental in representing transformations in 2D and 3D graphics, making it essential for rendering realistic images and animations.

**Machine Learning and Data Science**: Linear algebra plays a crucial role in machine learning and data science. Algorithms such as linear regression, support vector machines, and principal component analysis are based on linear algebra concepts. Matrices and vectors are used to represent and manipulate data, and linear transformations are employed in feature engineering and dimensionality reduction.

## INNER PRODUCT SPACE:

**An inner product space is a vector space where each pair of vectors has a defined "inner product", often denoted as $< u, v >$**

Given $u \in \mathbb{R}^n$ and v$\in \mathbb{R}^n$ we define the inner product / dot product of $u$ & $v$ as

$$u.v = < u, v > = u^T v = v^T u = u_1 v_1 + u_2 v_{2+\dots} + u_n v_n = < v, u >$$

Note: $< u, u > = \|u\|^2$ represents norm or length of the vector.

If v is a nonzero vector in $\mathbb{R}^n$, then the vector $\hat{u} = \frac{v}{\|v\|}$ , the vector u is called the unit vector in the direction of v. The process of finding the unit vector in the direction of v is called normalizing the vector v.

**GEOMETRIC INTERPRETATION:**

**1. Angle and length Preservation:**

- Inner product operation provides a way to measure angles between vectors and defines the notion of length or magnitude of vectors.
- Inner product allows you to determine the angle between vectors and ascertain whether they are orthogonal (at right angles) or aligned.

**2. Orthogonality:**

- Orthogonal vectors have an inner product of zero. Geometrically, this means they are perpendicular in space.

**NOTE:** This inner product function takes in two vectors from the space and produces a scalar value.

Example: Consider the vectors $u = (1,2,4), v = (2,-3,5)$ , $w = (4, 2, -3)$ in $R^3$ .
Find: $\langle u, v\rangle, \langle v, w\rangle$

Solution: $< u, v > = (1.2) + (2.-3) + (4.5) = 16$

$< v, w > = (2.4) + (-3.2) + (5.-3) = -13$

## PROJECTION AND ORTHOGONALITY:

Projection and orthogonality are two closely related concepts.

- Projection: The projection of a vector $u$ onto a subspace $W$ is the vector in $W$ that is closest to $u$. It is denoted by $Proj_w u$
- Orthogonality: Two vectors $u$ and $v$ are orthogonal if their inner product is zero. In other words, $< u, v >= 0$.

  Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space $V$, such that the **orthogonal projection of u onto v** is given by $Proj_v u = \frac{<u,v>}{<v,v>}\ v$

  If **v** is a initial vector, then $< v, v >= \|v\|^2 = 1$

  The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$Proj_v u =< u, v >\ v$$

**Example:** Use the Euclidean inner product in $R^3$ to find the orthogonal projection of

u = $(6, 2, 4)$ onto v= $(1, 2, 0)$.

Solution: Let $Proj_v u = \frac{<u,v>}{<v,v>}\ v$ ….. (1)

$$< u, v >= (6 \times 1) + (2 \times 2) + (4 \times 0) = 10$$

$$< v, v >= 1^2 + 2^2 + 0 = 5$$

Sub in (1), $Proj_v u = (2,4,0)$

So, the orthogonal projection of $u = (6,2,4)$, onto $v = (1,2,0)$ is $(2,4,0)$.