Module-5

LINEAR TRANSFORMATIONS

CONTENTS:

- Definition and examples,
- Matrix representations of a linear transformation.
- Change of coordinates and change of basis,
- Rank and nullity of a linear operator,
- Rank-nullity theorem,
- Eigenvalues and eigenvectors,
- Diagonalization.

LEARNING OBJECTIVES:

- Understand and articulate the definition of a linear transformation and its properties.
- Understand how matrices represent linear transformations and vice versa.
- Apply the methods for changing coordinates and basis using transition matrices.
- Apply the rank-nullity theorem to relate the dimensions of the domain, range, and kernel.
- Able to find eigenvalues and corresponding eigenvectors for given matrices.
- Diagonalize a given matrix by finding eigenvectors and constructing the diagonal matrix.

Here are some major applications:

1. Computer Science and Computer Graphics:

Computer Graphics: Matrices are used to represent transformations, such as rotation, translation, and scaling. Linear algebra is crucial for 3D graphics and animation.

Data Analysis: Techniques like Principal Component Analysis (PCA) use linear algebra for dimensionality reduction and feature extraction.

2. Physics:

Quantum Mechanics: Matrices and vectors are fundamental in representing quantum states and operations in quantum mechanics.

Mechanics and Dynamics: Linear algebra is used in the analysis of systems of linear equations representing physical systems.

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3. Engineering:

Linear transformations are used in circuit analysis and signal processing. For example, Fourier transforms and Laplace transforms are linear transformations that are essential in analyzing and designing electrical circuits.

4. Machine Learning and Data Science:

Regression Analysis: Linear regression is a common statistical method for predicting outcomes based on linear relationships between variables.

Neural Networks: The weights and biases in neural networks are often optimized using linear algebra operations.

5. Economics and Finance:

Input-Output Models: Linear algebra is used to model and solve input-output systems, describing economic relationships.

Portfolio Optimization: Linear algebra is applied in optimizing investment portfolios.

6. Statistics:

Multivariate Statistics: Covariance matrices, factor analysis, and multivariate regression rely on linear algebra concepts.

Markov Chains: Linear algebra is used to analyze and solve problems related to Markov processes.

7. Cryptography:

Public Key Cryptography: Algorithms like RSA use linear algebraic structures, such as modular arithmetic with matrices, for secure communication.

8. Biomedical Sciences:

Image Processing: Linear algebra is used in medical imaging for tasks like image reconstruction and feature extraction.

Population Modeling: Linear algebraic models are employed to analyze population dynamics in epidemiology.

9. Operations Research:

Linear Programming: Optimization problems, common in operations research, are often solved using linear algebra techniques.

10. Environmental Science:

Population Modeling: Linear algebra is used to model and analyze the growth and interaction of populations in ecological systems.

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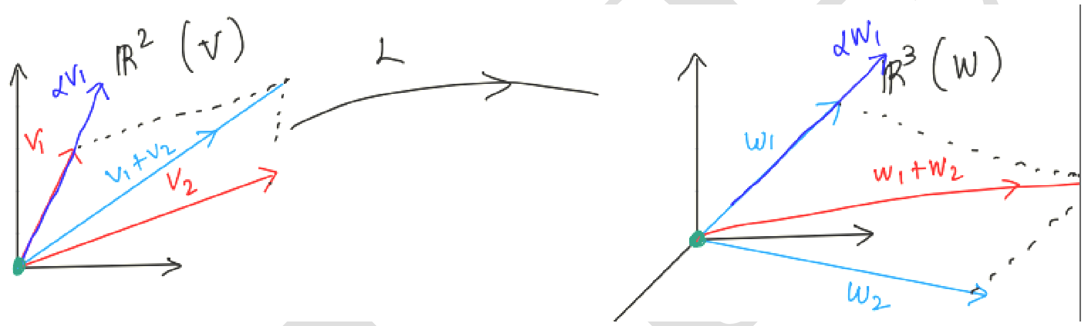
These applications demonstrate the broad impact and importance of linear algebra in various scientific, engineering, and economic disciplines. Understanding linear algebra is foundational for tackling complex problems in these fields.

DEFINITION

A mapping L from a vector space V into a vector space W, denoted by $L: V \to W$, is said to be a linear transformation if it satisfies two properties for all vectors v_1 and v_2 in the vector space V and all scalars c

1. Additivity: $T(v_1 + v_2) = T(v_1) + T(v_2)$

2. Scalar Multiplication: $T(cv_1) = cT(v_1)$

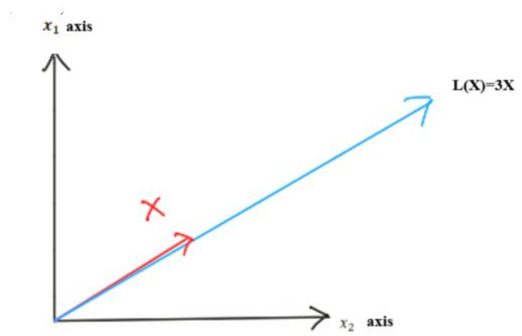


Note: If $L: V \to V$ we will refer to a linear transformation L as linear operator on V

That is, Linear operator is a linear transformation from a vector space to itself.

EXAMPLE OF A LINEAR TRANSFORMATION:

1**. Let L be the operator defined by $L(X) = 3X$ for each $X \in R^2$**
It is a linear operator because
i) $L(X + Y) = 3(X + Y) = 3X + 3Y = L(X) + L(Y)$
ii) $L(\propto X) = 3(\propto X) = (3 \propto)X = (\propto 3)X \ = \propto (3X) \ = \propto L(X)$



In general, the linear operator $L(X) = \propto X$ can be thought of as a stretching or shrinking by a factor of $\propto$.

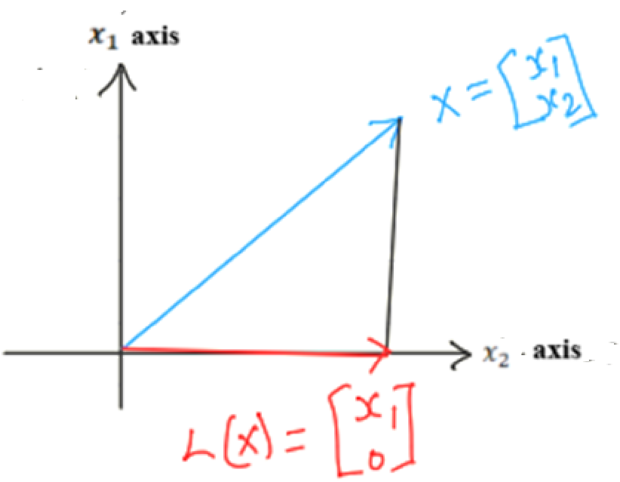
2. **Consider the mapping L defined by $L(X) = x_1 e_1$ for each $X \in R^2$. Thus , if $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $L(X) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.**

It is a Linear operator because for all $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and Y= $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, for all scalar $\propto$ and β,

$$L(\alpha X + \beta Y) = L\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ 0 \end{bmatrix} = (\alpha x_1 + \beta y_1) e_1$$

$$= \alpha(x_1 e_1) + \beta(y_1 e_1)$$

$$= \alpha L(X) + \beta L(X)$$



We can think L as a projection onto the x_1-axis

3. Suppose transformation $T: R^2 \to R^2$ defined as follows: $T(x, y) = (2x, -3y)$ verify that T is a linear transformation.

1. Additivity: For any two vectors (x_1, y_1) and (x_2, y_2) in R^2:

$T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = 2(x_1 + x_2) - 3(y_1 + y_2)$

$T(x_1, y_1) + T(x_2, y_2) = (2x_1, -3y_1) + (2x_2, -3y_2) = (2(x_1 + x_2), -3(y_1 + y_2))$

$T((x_1, y_1) + (x_2, y_2)) = T(x_1, y_1) + T(x_2, y_2)$

So the additivity property holds.

2. Scalar Multiplication: For any scalar c and vector (x, y) in R^2:

$$T\big(c(x,y)\big) = T\big((cx,cy)\big) = \big(2(cx), -3(cy)\big)$$

$$cT(x,y) = c(2x,-3y) = \big(2(cx), -3(cy)\big)$$

$$\therefore\ T\big(c(x,y)\big) = cT(x,y)$$

So the scalar multiplication property holds.

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION:

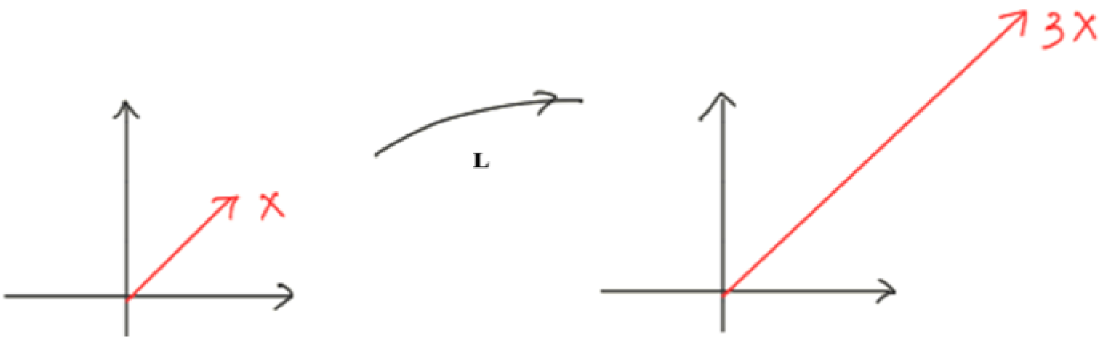
Let L be a linear transformation from R^n to R^m, there is matrix of order $m \times n$ such that

$L(X) = AX$ where $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}$ for any $X \in R^n$ and $L(e_i) = \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{mi} \end{bmatrix}$, i= $1,2,3, \dots, n$

Definition: If $X = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ is an arbitrary vector in R^n $L(X) = x_1L(e_1) + x_2L(e_2) + \cdots + x_nL(e_n)$ $= \big(L(e_1), L(e_2), \dots, L(e_n)\big)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ if $L(e_i) = \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{mi} \end{bmatrix}$ then $L(X) = AX$.

Example:

1. $L(X) = 3X$



Here $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, X= $x_1e_1 + x_2e_2$

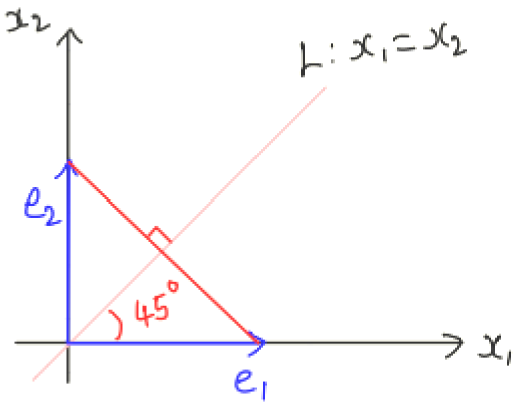
$$L(X) = L(x_1e_1 + x_2e_2) = x_1L(e_1) + x_2L(e_2)$$

And $AX = \big(L(e_1)\ L(e_2)\big)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ thus matrix representation of the given linear transformation is A=$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

2. Let $L(X) = x_1 e_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, L: R^2 \to R^2$ to find matrix representation A,

$$L(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad L(e_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Determine the matrix for transformation, projection onto 45$^\cdot$ line



L be the 45$^\cdot$ line projection of e_1=$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto L is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

PROBLEMS

1. Find the matrix of linear transformation $T: V_2(R) \to V_3(R)$ defined by $T(x, y) = (x + y, x, 3x - y)$ with respect to a standard basis.

Solution:

Let $\{e_1, e_2\}$ and $\{f_1, f_2, f_3\}$ be the standard basis of $V_2(R)$ and $V_3(R)$ respectively.

$$e_1 = (1,0), e_2 = (0,1)$$

$$f_1 = (1,0,0), f_2 = (0,1,0), f_3 = (0,0,1)$$

Now,

$$T(e_1) = T(1,0) = (1,1,3) = 1. f_1 + 1. f_2 + 3. f_3$$

$$T(e_2) = T(0,1) = (1,0,-1) = 1. f_1 + 0. f_2 - 1. f_3$$

$$\therefore \; A_T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

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2. If T is a mapping from $V_2(R)$ into $V_2(R)$ defined by $T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ Show that T is a linear transformation.

Solution:

Let $\propto = (x_1, x_2), \beta = (y_1, y_2)$ be two arbitrary elements of $V_2(R)$.

Consider $T(\propto + \beta) = T(x_1 + y_1, x_2 + y_2)$

$= [(x_1 + y_1)\cos\theta - (x_2 + y_2)\sin\theta, (x_1 + y_1)\sin\theta + (x_2 + y_2)\cos\theta]$

$= [(x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta), (x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta)]$

$= (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta) + (y_1\cos\theta - y_2\sin\theta, y_1\sin\theta + y_2\cos\theta)$
$= T(x_1, x_2) + T(y_1, y_2)$
$= T(\propto) + T(\beta)$
Let $c \in R$ be any scalar,
Consider
$T(c.\propto) = T(cx_1, cx_2)$
$= (cx_1\cos\theta - cx_2\sin\theta, cx_1\sin\theta + cx_2\cos\theta)$
$= c(x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta)$
$= cT(x_1, x_2) = cT(\propto)$
Hence T is a linear transformation.

PRACTICE PROBLEMS
3. If $T: V_1(R) \to V_2(R)$ **is defined by** $T(x) = (x, x^2, x^3)$ **Verify whether T is linear or not.**
4. If $T: 2(R) \to V_2(R)$ **defined by** $T(x,y) = (2x - y, x + 3y)$ **Verify whether T is linear or not.**

Some interesting transformation of $x \in \mathbb{R}^2$

i) $I_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$

This is called identity transformation.

Example:

$I_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = x$

ii) $R_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = y$

$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is called a reflection.

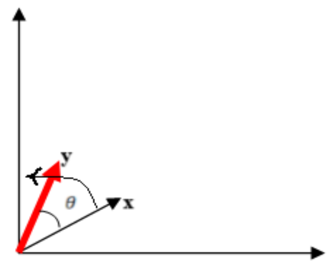
Example:

$$R_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = y$$

iii) $R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

$$R_{\theta x} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 cos\theta - x_2 sin\theta \\ x_1 sin\theta + x_2 cos\theta \end{bmatrix} = y$$

R_θ: Rotates the vector by an angle θ counter clockwise direction.



$$R_{(-\theta)} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

$R_{(-\theta)}$ is the inverse of R_θ

REMARK: A vector space has two operations defined on it, namely addition and scalar multiplication. Linear transformations between vector spaces are those transformations that preserve these linear structures in the following sense.

DEFINITION **Let U and V be vector spaces. Let u and v be vectors in U and let c be a scalar. A transformation $T: U \to V$ is said to be linear if**

$$\boldsymbol{T}(\mathbf{u} + \mathbf{v}) = \boldsymbol{T}(\mathbf{u}) + \boldsymbol{T}(\mathbf{v})$$
$$\boldsymbol{T}(\boldsymbol{c}\mathbf{u}) = \boldsymbol{cT}(\mathbf{u})$$

The first condition implies that T maps the sum of two vectors into the sum of the images of those vectors. The second condition implies that T maps the scalar multiple of a vector into the same scalar multiple of the image of that vector. Thus, the operations of addition and scalar multiplication are preserved under a linear transformation. We remind the reader of how to use the definition to check for linearity between two $\mathbf{R}^n$-type vector spaces with the following example. This leads to a discussion of more general vector spaces.

EXAMPLE 1 Prove that the following transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ is linear.

$$T(x, y) = (2x, x + y)$$

SOLUTION

We first show that T preserves addition. Let (x_1, y_1) and (x_2, y_2) be elements of $\mathbf{R}^2$. Then

$$\begin{aligned} T((x_1, y_1) + (x_2, y_2)) &= T(x_1 + x_2, y_1 + y_2) && \text{by vector addition} \\ &= (2x_1 + 2x_2, x_1 + x_2 + y_1 + y_2) && \text{by definition of } T \\ &= (2x_1, x_1 + y_1) + (2x_2, x_2 + y_2) && \text{by vector addition} \\ &= T(x_1, y_1) + T(x_2, y_2) && \text{by definition of } T \end{aligned}$$

Thus T preserves vector addition.

We now show that T preserves scalar multiplication. Let c be a scalar.

$$\begin{aligned} T(c(x_1, y_1)) &= T(cx_1, cy_1) && \text{by scalar multiplication of a vector} \\ &= (2cx_1, cx_1 + cy_1) && \text{by definition of } T \\ &= c(2x_1, x_1 + y_1) && \text{by scalar multiplication of a vector} \\ &= cT(x_1, y_1) && \text{by definition of } T \end{aligned}$$

Thus T preserves scalar multiplication. T is linear.

The following example illustrates a linear transformation between function vector spaces.

EXAMPLE 2 Let P_n be the vector space of real polynomial functions of degree $\leq n$.

Show that the following transformation $T: P_2 \to P_1$ is linear.

$$T(ax^2 + bx + c) = (a + b)x + c$$

SOLUTION

Let $ax^2 + bx + c$ and $px^2 + qx + r$ be arbitrary elements of P_2.

Then

$$\begin{aligned} &T((ax^2 + bx + c) + (px^2 + qx + r)) \\ &= T((a + p)x^2 + (b + q)x + (c + r)) \\ &= (a + p + b + q)x + (c + r) \quad \text{by definition of } T \\ &= (a + b)x + c + (p + q)x + r \\ &= T(ax^2 + bx + c) + T(px^2 + qx + r) \quad \text{by definition of } T \end{aligned}$$

Thus T preserves addition.

We now show that T preserves scalar multiplication. Let k be a scalar.

$$\begin{aligned} T\big(k(ax^2+bx+c)\big) &= T(kax^2+kbx+kc) \\ &= (ka+kb)x+kc \\ &= k\big((a+b)x+c\big) \\ &= kT(ax^2+bx+c) \qquad \text{by definition of } T \end{aligned}$$

by definition of T

T preserves scalar multiplication. Therefore, T is a linear transformation.

We now see that the properties of the derivative that the reader will have met in calculus courses imply that the derivative is a linear transformation.

EXAMPLE 3 Let D be the operation of taking the derivative. (D is the same as $\frac{d}{dx}$. It is a more appropriate notation in this context than $\frac{d}{dx}$.) D can be interpreted as a mapping of P_n into itself. For example,

$$D(4x^3-3x^2+2x+1) = 12x^2-6x+2$$

D maps the element $4x^3-3x^2+2x+1$ of P_3 into the element $12x^2-6x+2$ of P_3.

Let f and g be elements of P_n and c be a scalar. We know that a derivative has the following properties.

$$D(f+g) = Df + Dg$$
$$D(cf) = cD(f)$$

The derivative thus preserves addition and scalar multiplication of functions. It is a linear transformation.

Change of basis:

Let's consider a vector space V with two different bases: $B = \{v_1, v_2, \dots v_n\}$ and $B' = \{u_1, u_2, \dots u_n\}$

If we have a vector v expressed in terms of the basis B as $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, then we can find the representation of the same vector in terms of the basis B' by solving a system of linear equations.

Change of Basis Matrix:

The relationship between the two bases can be expressed using a matrix, known as the change of basis matrix. If P is the matrix whose columns are the vectors in B, and P′ is the matrix whose columns are the vectors in B′, then the change of basis matrix from B to B′ is given by $P'^{-1}P$. This matrix essentially transforms a vector from the original basis to a new basis.

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Given a vector v expressed in terms of the original basis B as $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, its representation in the new basis B′ is given by $v' = P'^{-1}v$, where v′ is the column vector of coefficients in the new basis.

Change of Basis in Linear Transformations:

Linear transformations can be represented by matrices. Suppose there is a linear transformation

$T: V \rightarrow W$ between vector spaces V and W. If V and W have bases B and B′ respectively, and P and P′ are the change of basis matrices, then the matrix representation of T in the B-B′ basis is given by $P'^{-1}TP$

Definition:

The standard basis for R^2 is $\{e_1, e_2\}$. Any vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in R^2 can be expressed as linear combination $X = x_1e_1 + x_2e_2$.

The scalars x_1 and x_2 are coordinates of X wrt $\{e_1, e_2\}$. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is called coordinate vector of X wrt $\{e_1, e_2\}$ or just coordinate vector of X.

Now let $\{y, z\}$ be any other basis in R^2. Then vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can also be represented uniquely as $X = x_1y + x_2z$

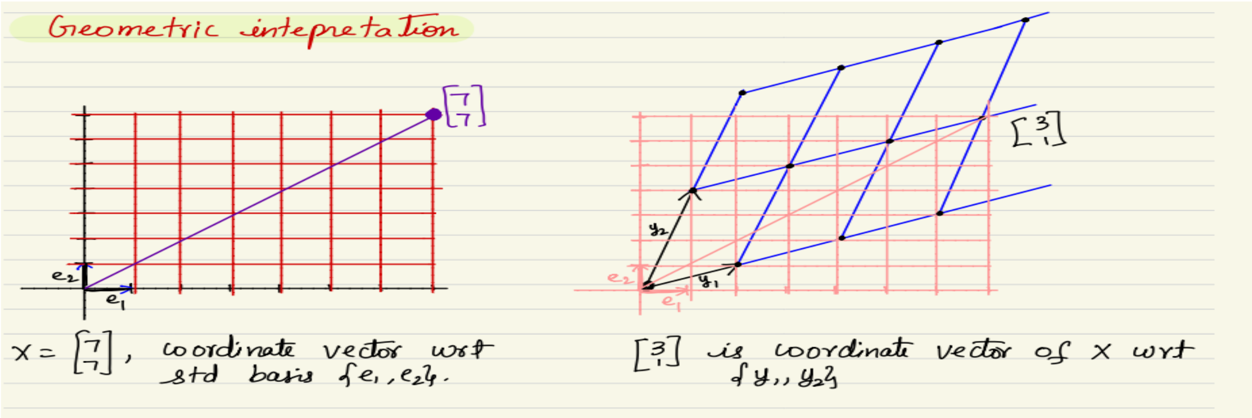
The scalars x_1 and x_2 are coordinates of X wrt $\{y, z\}$. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is called coordinate vector of X wrt $\{y, z\}$

The coordinate vector of X wrt the basis $\{z, y\}$ is $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ since $X = x_2z + x_1y$

ie coordinate vector changes when the order of basis changes. To avoid confusion, we consider ordered basis.

In ordered basis $\{y, z\}$, y is the 1st basis vector, z is the 2nd basis vector.

Example: Let $\{y_1, y_2\}$ be the basis in R^2, where $y_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $y_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Let $X = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ be a vector in R^2. Then coordinate vector of X wrt $\{y_1, y_2\}$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ sine $X = 3y_1 + 1y_2$



Changing coordinates:

1. Consider the following basis of R^2: $E = \{e_1, e_2\} = \{(1,0), (0,1)\}$ and $S = \{u_1, u_2\} = \{(1,3), (1,4)\}$

i) Find the change of basis matrix P from the matrix E to S

ii) Find the change of basis matrix Q from the matrix S to E

iii) Find the coordinate of $v = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ relative to S

Solution:

i) To find the change of basis matrix from basis E to basis S, you need to express each vector in basis E as a linear combination of vectors in basis S and then arrange the coefficients in a matrix. Let's denote the change of basis matrix from E to S as P.

Given basis vectors: $E = \{(1,0), (0,1)\}, S = \{(1,3), (1,4)\}$

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

For (1,0) in basis E, express it as a linear combination of basis vectors in S:

$(1,0) = p_{11}(1,3) + p_{21}(1,4)$

This leads to the system of equations:

$1 = p_{11} + p_{21}$ and $0 = 3p_{11} + 4p_{21}$

Solving this system will give you the values of $p_{11} = 4, p_{21} = -3$

Repeat the process for
$(0,1) = p_{12}(1,3) + p_{22}(1,4)$

This leads to the system of equations:

$0 = p_{12} + p_{22}$ and $1 = 3p_{21} + 4p_{22}$

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Solving this system will give you the values of $p_{21} = -1, p_{22} = 1$

The resulting matrix P will be the change of basis matrix from E to S.

$$P = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

ii) Now, to find the change of basis matrix from S to E, you can follow a similar procedure. Let $q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix}$

For (1,3) in basis S, express it as a linear combination of basis vectors in E:

$$(1,3) = q_{11}(1,0) + q_{12}(0,1)$$

This leads to the system of equations: $1 = q_{11}$ and $3 = q_{12}$ which gives the values of $q_{11}$and q_{12}

Repeat the process for $(1,4) = q_{21}(1,0) + q_{22}(0,1)$

This leads to the system of equations:$1 = q_{21}$and $4 = q_{22}$ gives the values of $q_{21}$and q_{22}

The resulting matrix Q will be the ch1ange of basis matrix from S to E.

$$q = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

iii) To find the coordinates of the vector $v = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ with respect to the basis $S = \{u_1, u_2\} = \{(1,3),(1,4)\}$, we need to express v as a linear combination of $u_1 and\ u_2$. The coordinates [a,b] of v with respect to the basis S are the coefficients in this linear combination. The given basis vectors are: $u_1 = (1,3)$ nd $u_2 = (1,4)$, we want to find a and b such that: $v = au_1 + bu_2$

Substitute the components of v and the basis vectors: $\begin{bmatrix} 5 \\ -3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Now, set up a system of equations and solve for a and b: $a + b = 5;\ 3a + 4b = -3$

On solving $a = 23, b = -18$

2. The vector $u_1 = (1,2,0), u_2 = (1,3,2,), u_3 = (0,1,3)$form a basis S of R^3

i) Find the change of basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of R^3 to the basis S

ii) Find the change of basis matrix Q from the basis S to the usual basis E

Solution: P= $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}, Q = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$

3. Find the coordinates of the following vectors relative to the basis $(1,1,2),(2,2,1),(1,2,2)$

i) (1,1,1) **Ans:** (1/3,1/3,0)

ii) (1,0,1) **Ans:** (4/3,1/3,-1)

iii) (1,1,0) **Ans:** (-1/3,2/3,0)

Problems:

1. Let $V = R^3$ and let $S = \{v_1, v_2, v_3\}$ *and* $T = \{w_1, w_2, w_3\}$ be the basis of R^3, where $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, v_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}, w_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, w_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, w_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Obtain the transition matrix from T to S

Solution: $\begin{bmatrix}\frac{3}{2} & 0 & 1\\-2 & 1 & -1\\-\frac{9}{2} & 2 & -2\end{bmatrix}$

2. Let $V = R^3$ and let $E = \{v_1, v_2, v_3\}$ *and* $F = \{u_1, u_2, u_3\}$ be the basis of R^3, where $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, v_2 = \begin{bmatrix}2\\3\\2\end{bmatrix}, v_3 = \begin{bmatrix}1\\5\\4\end{bmatrix}, u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, u_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}, u_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$

Let $X = 3v_1 + 2v_2 - v_3; Y = v_1 - 3v_2 + 2v_3$. Obtain the transition matrix from E to F, use it to find coordinate vectors of X and Y with respect to the ordered basis F

Solution: Transition matrix = $\begin{bmatrix}1 & 1 & -3\\-1 & -1 & 0\\1 & 2 & 4\end{bmatrix}$, $[X]_F = \begin{bmatrix}8\\-5\\3\end{bmatrix}$, $[Y]_F = \begin{bmatrix}-8\\2\\3\end{bmatrix}$

3. Find the transition matrix corresponding to the change of basis from $\{v_1, v_2\}$ to $\{u_1, u_2\}$, where $v_1 = \begin{bmatrix}5\\2\end{bmatrix}, v_2 = \begin{bmatrix}7\\3\end{bmatrix}, u_1 = \begin{bmatrix}3\\2\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\end{bmatrix}$

Solution: $\begin{bmatrix}3 & 4\\-4 & -5\end{bmatrix}$

4. Let $u_1 = \begin{bmatrix}1\\-1\end{bmatrix}, u_2 = \begin{bmatrix}-2\\3\end{bmatrix}$. Find the transition matrix from $\{e_1, e_2\}$ to $\{u_1, u_2\}$ nd determine the coordinate of $X = \begin{bmatrix}1\\2\end{bmatrix}$ with respect to $\{u_1, u_2\}$

Solution: Transition matrix = $\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$, $[X]_{\{u_1,u_2\}} = \begin{bmatrix}7\\3\end{bmatrix}$

5. Let $u_1 = \begin{bmatrix}3\\2\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\end{bmatrix}, X = \begin{bmatrix}7\\4\end{bmatrix}$. Find the coordinate of X with respect to $\{u_1, u_2\}$

Solution: $[X]_{\{u_1,u_2\}} = \begin{bmatrix}3\\-2\end{bmatrix}$

APPLICATIONS OF CHANGE OF BASIS:

Changing the basis of a vector space involves expressing vectors in terms of different sets of basis vectors. This concept finds applications in various fields:

Computer Graphics: In computer graphics, objects are represented in 3D space using vectors. Changing the basis allows for transformations like rotation, scaling, and translation of objects. It's crucial for rendering objects in different orientations or coordinate systems.

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Engineering (Control Systems): Control systems analysis often involves transforming differential equations or system representations from one basis to another to simplify analysis and design procedures.

Signal Processing: Signal processing involves manipulating signals using various mathematical operations. Changing the basis can help simplify signal analysis, compression, and filtering by choosing bases that better represent signal characteristics.

Machine Learning and Data Analysis: Techniques like Principal Component Analysis (PCA) involve changing the basis of data vectors to find more meaningful representations or reduce the dimensionality of high-dimensional data.

Cryptography: Changing bases can be used in cryptographic algorithms, where secure communications involve transforming data using specific bases to make it harder for unauthorized parties to interpret.

Linear Transformation Analysis: Understanding linear transformations involves studying how they change bases and how this affects the representation of vectors and matrices. Applications include understanding transformation properties in various fields, including physics and engineering.

In essence, changing the basis provides a way to view data or systems from different perspectives, simplifying calculations or analyses, highlighting important features, and aiding in problem-solving in various disciplines. It's a fundamental tool in manipulating and understanding vector spaces and their applications across diverse fields of study.

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Kernel and Range

We know that a linear transformation is a function from one vector space (called the domain) into another vector space (called the codomain.) There are two further vector spaces called kernel and range that are associated with every linear transformation. In this section we introduce and discuss the properties of these spaces.

The following theorem gives an important property of all linear transformations. It paves the way for the introduction of kernel and range.

THEOREM

Let $T: U \to V$ be a linear transformation. Let $\mathbf{0}_U$ and $\mathbf{0}_V$ be the zero vectors of U and V. Then

$$T(\mathbf{0}_U) = \mathbf{0}_V$$

That is, a linear transformation maps a zero vector into a zero vector.

Proof: Let $\mathbf{u}$ be a vector in U and let $T(\mathbf{u}) = \mathbf{v}$. Let 0 be the zero scalar. Since $0\mathbf{u} = \mathbf{0}_U$ and $0\mathbf{v} = \mathbf{0}_V$ and T is linear, we get

$$T(\mathbf{0}_U) = T(0\mathbf{u}) = 0T(\mathbf{u}) = 0\mathbf{v} = \mathbf{0}_V$$

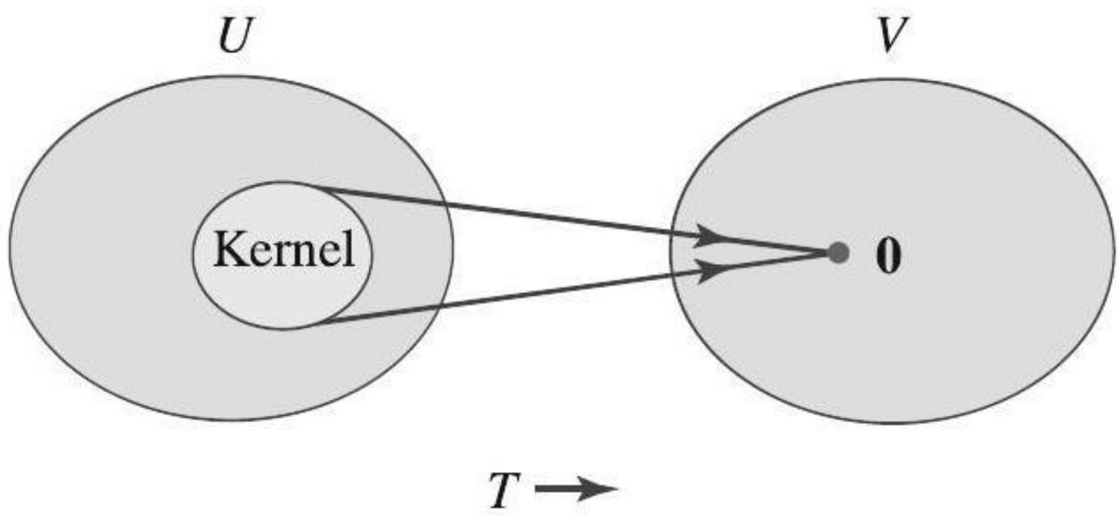
For example, $T(x, y, z) = (3x, y + z)$ is a linear transformation of $\mathbf{R}^3 \to \mathbf{R}^2$. The zero vector of $\mathbf{R}^3$ is $(0,0,0)$ and the zero vector of $\mathbf{R}^2$ is $(0,0)$. We see that $T(0,0,0) = (0,0)$.

DEFINITION Let $T: U \to V$ be a linear transformation.

The set of vectors in U that are mapped into the zero vector of V is called the kernel of T. The kernel is denoted ker(T). The kernel is often called **the null space** and denoted null (T).

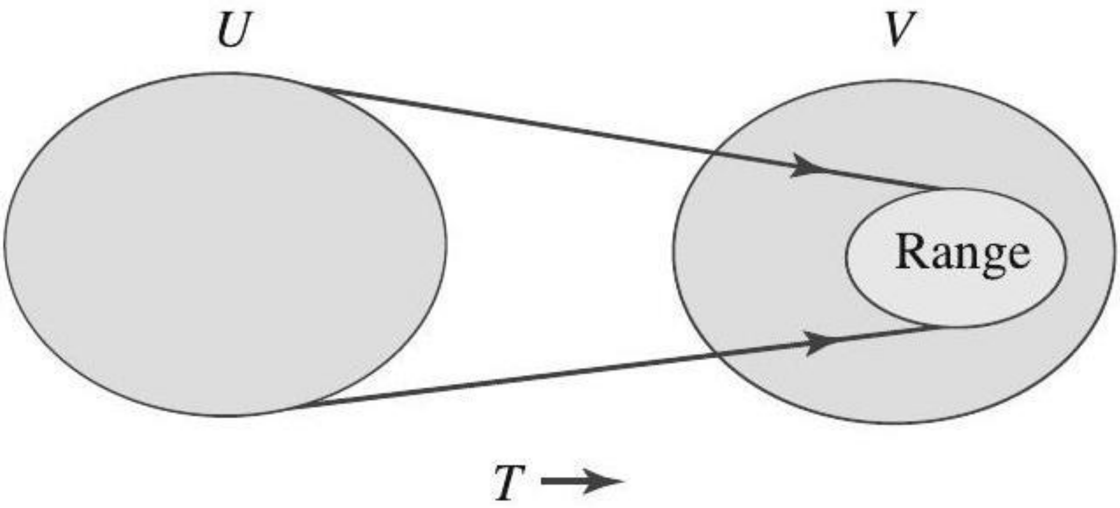
The set of vectors in V that are the images of vectors in U is called the range of T. The range is denoted range (T).

We illustrate these sets in following Figure. Whenever we introduce sets in linear algebra, we are interested in knowing whether they are vector spaces or not. We now find that the kernel and range are indeed vector spaces.



Kernel

All vectors in U that are mapped into $\mathbf{0}$



Range

All vectors in V that are images of vectors in U

THEOREM

Let $T: U \to V$ be a linear transformation.

(a) The kernel of T is a subspace of U.

(b) The range of T is a subspace of V.

Proof

(a) From the previous theorem, we know that the kernel is nonempty since it contains the zero vector of U. To prove that the kernel is a subspace of U, it remains to show that it is closed under addition and scalar multiplication.

First, we prove closure under addition. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be elements of $\ker(T)$. Thus $T(\mathbf{u}_1) = \mathbf{0}$ and $T(\mathbf{u}_2) = \mathbf{0}$. Using the linearity of T, we get

$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

The vector $\mathbf{u}_1 + \mathbf{u}_2$ is mapped into $\mathbf{0}$. Thus $\mathbf{u}_1 + \mathbf{u}_2$ is in $\ker(T)$.

Let us now show that $\ker(T)$ is closed under scalar multiplication. Let c be a scalar. Again, using the linearity of T, we get

$$T(c\mathbf{u}_1) = cT(\mathbf{u}_1) = c\mathbf{0} = \mathbf{0}$$

The vector $c\mathbf{u}_1$ is mapped into $\mathbf{0}$. Thus $c\mathbf{u}_1$ is in $\ker(T)$.

MITE

The kernel is closed under addition and under scalar multiplication. It is a subspace of U.

(b) The previous theorem tells us that the range is nonempty since it contains the zero vector of V. To prove that the range is a subspace of V, it remains to show that it is closed under addition and scalar multiplication. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be elements of range(T). Thus, there exist vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the domain U such that

$$T(\mathbf{w}_1) = \mathbf{v}_1 \text{ and } T(\mathbf{w}_2) = \mathbf{v}_2$$

Using the linearity of T,

$$T(\mathbf{w}_1 + \mathbf{w}_2) = T(\mathbf{w}_1) + T(\mathbf{w}_2) = \mathbf{v}_1 + \mathbf{v}_2$$

The vector $\mathbf{v}_1 + \mathbf{v}_2$ is the image of $\mathbf{w}_1 + \mathbf{w}_2$. Thus $\mathbf{v}_1 + \mathbf{v}_2$ is in the range.

Let c be a scalar. By the linearity of T,

$$T(c\mathbf{w}_1) = cT(\mathbf{w}_1) = c\mathbf{v}_1$$

The vector $c\mathbf{v}_1$ is the image of $c\mathbf{w}_1$. Thus $c\mathbf{v}_1$ is in the range.

The range is closed under addition and under scalar multiplication. It is a subspace of V.

EXAMPLE 4 **Find the kernel and range of the linear operator**

$$\boldsymbol{T(x, y, z) = (x, y, 0)}$$

SOLUTION

Since the linear operator T maps $\mathbf{R}^3$ into $\mathbf{R}^3$, the kernel and range will both be subspaces of $\mathbf{R}^3$.

Kernel: $\ker(T)$ is the subset that is mapped into $(0,0,0)$. We see that

$$\begin{aligned} T(x, y, z) &= (x, y, 0) \\ &= (0,0,0), \text{ if } x = 0, y = 0 \end{aligned}$$

Thus $\ker(T)$ is the set of all vectors of the form $(0,0,z)$. We express this

$$\ker(T) = \{(0,0,z)\}$$

Geometrically, $\ker(T)$ is the set of all vectors that lie on the z-axis.

Range: The range of T is the set of all vectors of the form $(x, y, 0)$. Thus

$$\text{range}(T) = \{(x, y, 0)\}$$

range (T) is the set of all vectors that lie in the xy-plane.

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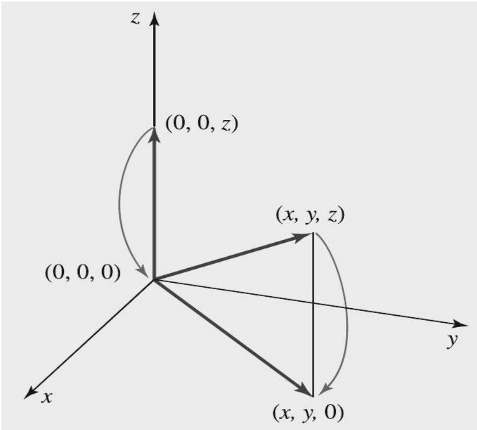
We illustrate this transformation in Figure. Observe that T projects the vector (x, y, z) into the vector $(x, y, 0)$ in the xy-plane. T projects all vectors onto the xy-plane. T is an example of a projection operator.



projection
$T(x, y, z) = (x, y, 0)$

Projections are important in applications. The world in which we live has three spatial dimensions. When we observe an object, however, we get a two-dimensional impression of that object, the view changing from location to location. Projections can be used to illustrate what three-dimensional objects look like from various locations. Such transformations, for example, are used in architecture, the auto industry, and the aerospace industry. The outline of the object of interest, relative to a suitable coordinate system, is fed into a computer. The computer program contains an appropriate projection transformation that maps the object onto a plane. The output gives a two-dimensional view of the object, the picture being graphed by the computer. In this manner various transformations can be used to lead to various perspectives of an object. We now discuss kernels and ranges of matrix transformations. The following theorem gives us information about the range of a matrix transformation.

THEOREM

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be defined by $T(\mathbf{u}) = A\mathbf{u}$. The range of T is spanned by the column vectors of A.

Proof: Let $\mathbf{v}$ be a vector in the range. There exists a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{v}$. Express $\mathbf{u}$ in terms of the standard basis of $\mathbf{R}^n$.

$$\mathbf{u} = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$$

Thus

$$\begin{aligned}\mathbf{v} &= T(c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n) \\ &= c_1T(\mathbf{e}_1) + \cdots + c_nT(\mathbf{e}_n)\end{aligned}$$

Therefore, the column vectors of A, namely, $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$, span the range of T.

EXAMPLE 5 Determine the kernel and the range of the transformation defined by the following matrix.

MITE

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

SOLUTION

A is a 3×3 matrix. Thus A defines a linear operator $T: \mathbf{R}^3 \to \mathbf{R}^3$,

$$T(\mathbf{x}) = A\mathbf{x}$$

The elements of $\mathbf{R}^3$ are written in column matrix form for the purpose of matrix multiplication.

Kernel: The kernel will consist of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ such that

$$T(\mathbf{x}) = \mathbf{0}$$

Thus,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This matrix equation corresponds to the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_2 + x_3 &= 0 \\ x_1 + x_2 + 4x_3 &= 0 \end{aligned}$$

On solving this system, we get many solutions, $x_1 = -5r, x_2 = r, x_3 = r$. The kernel is thus the set of vectors of the form $(-5r, r, r)$.

$$\text{Ker}(T) = \{(-5r, r, r)\}$$

$\text{Ker}(T)$ is a one-dimensional subspace of $\mathbf{R}^3$ with basis $(-5,1,1)$.

Range: The range is spanned by the column vectors of A. Write these column vectors as rows of a matrix and compute an echelon form of the matrix. The nonzero row vectors will give a basis for the range. We get

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors $(1,0,1)$ and $(0,1,1)$ span the range of T. An arbitrary vector in the range will be a linear combination of these vectors,

$$s(1,0,1) + t(0,1,1)$$

Thus, the range of T is

$$\text{Range}(T) = \{(s, t, s+t)\}$$

The vectors $(1,0,1)$ and $(0,1,1)$ are also linearly independent. Range (T) is a twodimensional subspace of $\mathbf{R}^3$ with basis $\{(1,0,1), (0,1,1)\}$.

The following theorem gives an important relationship between the "sizes" of the kernel and the range of a linear transformation.

THEOREM

Let $T: U \to V$ be a linear transformation. Then

$$\dim \ker(T) + \dim \text{range}(T) = \dim \text{domain}(T)$$

(Observe that this result holds for the linear transformation T of the previous example: $\dim \ker(T) = 1, \dim \text{range}(T) = 2, \dim \text{domain}(T) = 3$.)

Proof: (For reference) Let us assume that the kernel consists of more than the zero vector, and that it is not the whole of U.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be a basis for $\ker(T)$. Add vectors $\mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$ to this set to get a basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ for U. We shall show that $T(\mathbf{u}_{m+1}), \ldots, T(\mathbf{u}_n)$ form a basis for the range, thus proving the theorem.

Let $\mathbf{u}$ be a vector in U. $\mathbf{u}$ can be expressed as a linear combination of the basis vectors as follows.

$$\mathbf{u} = a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m + a_{m+1}\mathbf{u}_{m+1} + \cdots + a_n\mathbf{u}_n$$

Thus

$$T(\mathbf{u}) = T(a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m + a_{m+1}\mathbf{u}_{m+1} + \cdots + a_n\mathbf{u}_n)$$

The linearity of T gives

$$T(\mathbf{u}) = a_1T(\mathbf{u}_1) + \cdots + a_mT(\mathbf{u}_m) + a_{m+1}T(\mathbf{u}_{m+1}) + \cdots + a_nT(\mathbf{u}_n)$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are in the kernel, this reduces to

$$T(\mathbf{u}) = a_{m+1}T(\mathbf{u}_{m+1}) + \cdots + a_nT(\mathbf{u}_n)$$

$T(\mathbf{u})$ represents an arbitrary vector in the range of T. Thus, the vectors $T(\mathbf{u}_{m+1}), \ldots, T(\mathbf{u}_n)$ span the range.

It remains to prove that these vectors are also linearly independent. Consider the identity

$$b_{m+1}T(\mathbf{u}_{m+1}) + \cdots + b_nT(\mathbf{u}_n) = \mathbf{0}$$

where the scalars have been labeled thus for convenience. The linearity of T implies that

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$$T(b_{m+1}\mathbf{u}_{m+1} + \cdots + b_n\mathbf{u}_n) = \mathbf{0}$$

This means that the vector $b_{m+1}\mathbf{u}_{m+1} + \cdots + b_n\mathbf{u}_n$ is in the kernel. Thus, it can be expressed as a linear combination of the basis of the kernel. Let

$$b_{m+1}\mathbf{u}_{m+1} + \cdots + b_n\mathbf{u}_n = c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m$$

Thus

$$c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m - b_{m+1}\mathbf{u}_{m+1} - \cdots - b_n\mathbf{u}_n = \mathbf{0}$$

Since the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$ are a basis, they are linearly independent. Therefore, the coefficients are all zero.

$$c_1 = 0, \ldots, c_m = 0, b_{m+1} = 0, \ldots, b_n = 0$$

Returning to identity (1), this implies that $T(\mathbf{u}_{m+1}), \ldots, T(\mathbf{u}_n)$ are linearly independent. The set of vectors $T(\mathbf{u}_{m+1}), \ldots, T(\mathbf{u}_n)$ is a basis for the range. The theorem is proven.

Note that the "bigger" the kernel, the "smaller" the range, and vice versa. The rank (r) of a matrix is the number of leading variables in Row echelon form. The nullity of a matrix is the number of free variables in Row echelon form i.e., (n-r).

Rank Nullity Theorem

Let A be an $m \times n$ matrix that defines a linear transformation T. We have mentioned that the kernel of T is also called the null space of T, denoted null(T). The dimension of null (T) is called nullity (T). The range of T is the subspace of $\mathbf{R}^m$ spanned by the column vectors of A). Thus, the dimension of the range of T is rank(A); and we say that rank(T) = rank(A). Then we can write

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim \operatorname{domain}(T)$$

EXAMPLE 6 Consider the following matrix A and the linear transformation T defined by A. Find rank(A), dim domain(T), dim range(T), dim ker(T), and nullity(T).

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

SOLUTION

Observe that the matrix A is in echelon form. The nonzero row vectors are linearly independent. Thus, rank(A) = 2. Since A is a 3×3 matrix, $T: \mathbf{R}^3 \to \mathbf{R}^3$. Thus dim domain($T$) = 3.

$$\dim \operatorname{range}(T) = \operatorname{rank}(A) = 2.$$

$$\dim \ker(T) = \dim \operatorname{domain}(T) - \dim \operatorname{range}(T) = 3 - 2 = 1$$

$$\therefore \text{nullity}(T) = \dim \text{kernel}(T) = 1.$$

7. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$$

Here, the rank is 1, since the basis {(3, −6), (1, −2)} can be reduced to {(1, −2)}. The kernel of A is vectors such that $Av = 0$, which is a vector space spanned by {(1, −3)} and has dimension 1. Hence the rank and nullity are both 1, and sum to 2, the number of columns in A.

8.This can be applied to non-square matrices as well. For instance, in the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 4 & 2 \end{bmatrix}$$

the rank is 2, spanned by the first two columns of A, and the kernel is a vector space spanned by {(22, −7, 3)} that thus has dimension 1. As expected, the sum of the rank and nullity is thus 3, the number of columns in A.

9. $T: V_3(R) \to V_3(R)$ defined by $T(x, y, z) = (x + y, x - y, 2x + z)$, verify the rank nullity theorem.

Solution: $T(1,0,0) = (1,1,2): T(0,1,0) = (1,-1,0): T(0,0,1) = (0,0,1)$

Consider, $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_2 \to -\frac{1}{2(R_2)}} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore r(T) = 3 \; and \; n(T) = 0$$

$$r(T) + n(T) = 3 + 0 = 3 = d[V_3(R)]$$

Theorem is verified.

10. $T: V_3(R) \to V_3(R)$ defined by $T(x, y, z) = (x + 2y + z, z - x, y + z)$, verify the rank nullity theorem.

Answer:

$$r(T) + n(T) = 2 + 1 = 3 = d[V_3(R)]$$

11. $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix}$, find a basis for null-space(A) and verify rank nullity theorem.

Answer: rank(A) + nullity(A) = 2 + 2 = 4 = n.

12. determine the null space and verify the Rank-Nullity Theorem.

a. $A = \begin{bmatrix} 1 & 0 & -6 & -1 \end{bmatrix}$ b. $B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ c. $C = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 4 \\ 1 & 1 & 0 \end{bmatrix}$ d. $D = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 9 & -1 & 7 \\ 2 & 8 & -2 & 6 \end{bmatrix}$.

APPLICATIONS OF RANK AND NULLITY

The rank and nullity of a matrix are crucial concepts in linear algebra and have numerous applications across various fields:

Solving Systems of Linear Equations: Understanding the rank and nullity of a matrix is essential in solving systems of linear equations. The rank gives insights into the number of independent equations or rows in a system, while the nullity provides information about the dimension of the solution space.

Control Systems: In control theory, the rank-nullity theorem helps in analyzing the controllability and observability of systems. It assesses whether a system's dynamics can be controlled or observed given certain constraints.

Data Analysis (Statistics and Machine Learning): Understanding the rank and nullity of a data matrix is crucial in principal component analysis (PCA) and feature selection. The rank helps identify the dimensionality of the dataset, and the nullity can indicate redundant or irrelevant features.

Image and Signal Processing: In image and signal compression, the rank-nullity theorem aids in understanding the fundamental limits of compression algorithms. It helps in identifying the essential information in an image or signal that needs to be retained for faithful reconstruction.

Coding Theory: In coding theory, understanding the rank and nullity is crucial for error-correcting codes. The rank-nullity theorem is used to design codes that can detect and correct errors in data transmission.

Graph Theory: The rank and nullity of a matrix representation of a graph are used to analyze properties of graphs, such as connectivity, cycles, and paths.

Eigenvalues and Eigenvectors: The rank-nullity theorem is related to eigenvalues and eigenvectors. It helps in understanding the properties of eigenvalues, especially in determining the number of non-zero eigenvalues.

Physics (Quantum Mechanics): In quantum mechanics, matrices representing physical systems can have specific properties related to their rank and nullity, providing insights into the nature of the system's states and observables.

In essence, the rank and nullity of matrices play a fundamental role in various mathematical, engineering, scientific, and computational applications, providing crucial insights into the structure, behavior, and solvability of systems described by linear transformations or matrices.

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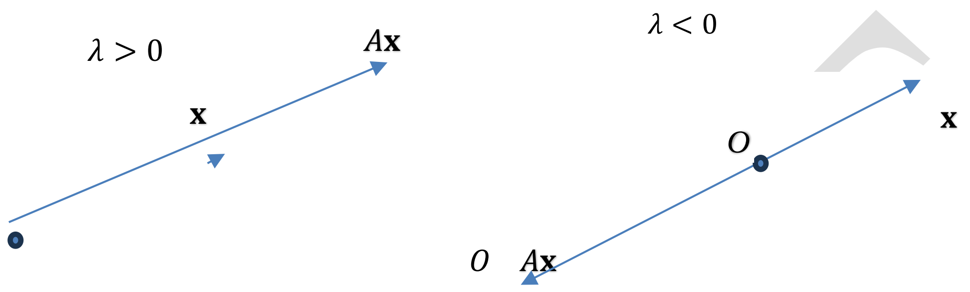
Eigen Values and Eigen Vectors: Let A be an $n \times n$ matrix. A Scalar λ is called an eigenvalue of A if there exist a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called eigenvector corresponding to λ.

Let us look at the geometrical significance of an eigenvector that corresponds to a nonzero eigenvalue. The vector $A\mathbf{x}$ is in the same or opposite direction as $\mathbf{x}$, depending on the sign of λ. An eigenvector of A is thus a vector whose direction is unchanged or reversed when multiplied by A.



x is an eigenvector of A. A**x** is in the same or opposite direction as **x.**

Computation of Eigenvalues and Eigenvectors: Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Thus $A\mathbf{x} = \lambda\mathbf{x}$.This equation may be rewritten,

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$
$$(A\mathbf{x} - \lambda I_n)\mathbf{x} = \mathbf{0}$$

This matrix equation represents a system of homogeneous linear equations having matrix of coefficients

$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ is a solution to this system. However, eigenvectors have been defined to be nonzero vectors. Further, nonzero solutions to this system of equations can only exist if the matrix of coefficients is singular, $|A - \lambda I_n| = 0$. Hence, solving the equation $|A - \lambda I_n| = 0$ for λ leads to all the eigenvalues of A.

On expanding the determinant $|A - \lambda I_n|$ we get a polynomial of degree n in λ. This polynomial is called the **characteristic polynomial** of A. The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation** of A. This characteristic equation will have n roots, some possibly repeated, some possibly complex numbers.

Thus an $n \times n$ matrix will have n eigenvalues, some of which may be repeated, and some of which may be complex numbers. The eigenvalues are then substituted back into the equation $(A\mathbf{x} - \lambda I_n)\mathbf{x} = \mathbf{0}$ to find the corresponding eigenvectors.

Example 1: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$

Solution: Let us first derive the characteristic polynomial of A. We get,

$$A - \lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

$$|A = \lambda I_2| = (-4-\lambda)(5-\lambda) + 18 = \lambda^2 - \lambda - 2$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2 \; or \; -1.$$

The eigenvalues of A are 2 or -1.

The corresponding eigenvectors are found by using these values of in the equation $(A\mathbf{x} - \lambda I_n)\mathbf{x} = \mathbf{0}$. There are many eigenvectors corresponding to each eigenvalue.

When $\lambda = 2$, we solve the equation $(A - 2I_2)x = 0$ for x . The matrix $(A - 2I_2)$is obtained by subtracting 2 from the diagonal elements of A. We get,

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This leads to the system of equation, $-6x_1 - 6x_2 = 0$

$$3x_1 + 3x_2 = 0$$

giving $x_1 = -x_2$. The solutions to this system of equations are $x_1 = -r, x_2 = r$, where r is a scalar. Thus the eigenvectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$r\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$$

When $\lambda = -1$, we solve the equation $(A + 1I_2)x = 0$ for x . The matrix $(A + 1I_2)$is obtained by subtracting 1 from the diagonal elements of A. We get,

$$\begin{bmatrix} -3 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This leads to the system of equation, $-3x_1 - 6x_2 = 0$

$$3x_1 + 3x_2 = 0$$

giving $x_1 = -2x_2$. The solutions to this system of equations are $x_1 = -2s, x_2 = s$, where s is a scalar. Thus, the eigenvectors of A corresponding to $\lambda = -1$ are nonzero vectors of the form,

$$s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$$

Example 2: Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

Solution:

We first obtain the matrix $[A - \lambda I]$.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

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The characteristic equation is $|A - \lambda I| = 0$.

Using row and column operations to simplify determinant, we obtain,

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$\text{OR } |A-\lambda I| = -(\lambda-10)(\lambda-1)^2$$

Solving the equation, $-(\lambda - 10)(\lambda - 1)^2 = 0$, gives the eigenvalues as $\lambda = 10 OR \lambda = 1 (REPEATED)$

To find the corresponding eigenvectors, we use the equation [A- $\lambda I]X = 0$.

For $\lambda = 10$,

$$[\text{A-} \lambda I]X = [A - 10I]X = 0$$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -5 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving the system of linear equations, x=2k, y=2k, z=k, where k is any scalar.

So, eigenvector of $\lambda = 10$ are non-zero vectors of the form $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The eigenspace is $\left\{ k \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$. The set $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ forms a basis and hence, the dimension is 1.

For $\lambda = 1$,

$$[A - \lambda I]X = [A - 1I]X = 0$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving the system of linear equations, $x = -s - t, y = s, z = 2t$, where s and t is any Scalars.

So, eigenvector of $\lambda = 1$ are non-zero vectors of the form $\begin{bmatrix} -s-t \\ 2s \\ 2t \end{bmatrix}$.

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Practice Problems:

Determine the eigen value and eigen vectors of the matrices,

a) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$

f) $\begin{bmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ g) $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$ h) $\begin{bmatrix} 5 & -2 & -2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{bmatrix}$

i) A company has three divisions: A, B, and C. The company's profit matrix is as follows:

$P = \begin{bmatrix} 10 & 2 & 3 \\ 3 & 8 & 5 \\ 4 & 6 & 7 \end{bmatrix}$ where P_{ij} represents the profit that the company makes from selling products from division i to division j. Determine the eigenvalues and eigenvectors of the profit matrix.

j) A college has three departments: Computer Science (CS), Electrical Engineering (EE), and Mechanical Engineering (ME). The college's admissions data for the past three years is shown in the following matrix:

$A = \begin{bmatrix} 100 & 80 & 70 \\ 90 & 70 & 60 \\ 80 & 60 & 50 \end{bmatrix}$ where the element A_{ij} represents the number of students who transferred from department i to department j in the past three years. Determine the eigenvalues and eigenvectors of the admissions matrix.

Diagonalization: A square matrix A is said to be **diagonalizable** if there exists a matrix P such that $D = P^{-1}AP$ a diagonal matrix. Where P is a matrix whose column matrix are linearly independent eigenvectors. The diagonal elements of D are the eigenvalues of A.

Remark:

1) If A is diagonalizable, then the column vectors of the diagonalizing matrix P are eigenvectors of A and the diagonal elements of D are the corresponding eigenvalues of A.

2) The diagonalizing matrix P is not unique. Reordering the columns of a given diagonalizing matrix P or multiplying them by nonzero scalars will produce a new diagonalizing matrix.

3) If A is $n \times n$ and A has n distinct eigenvalues, then A is diagonalizable. If the eigenvalues are not distinct, then A may or may not be diagonalizable depending on whether A has n linearly independent eigenvectors.

4) If A is diagonalizable, then Acan be factored into a product.

Example:

a) Show that the following matrix A is diagonalizable.

b) Find a diagonal matrix D that is similar to A.

Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -4$ and the corresponding eigenvectors are $p_1 = [3 \quad 1]^T$ and $p_2 = [1 \quad 2]^T$ Then $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$

It follows that, $P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= D$$

And $PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} = A$