# MODULE-2
# PARTIAL DIFFERENTIAL EQUATIONS

## Contents:

Solution of PDE's using Finite difference method

- Introduction to Second order PDE.
- General form of second order PDE.
- Classification of second order PDE.
- Solution of one-dimensional heat equation using Schmidt method.
- One dimensional wave equation using explicit method.
- Solution of two-dimensional Laplace equation.

**(RBT Levels: L1, L2 & L3)**

## Learning Objectives:

- This course aims to develop a systematic understanding of partial differential equations and enhance the application of these equations in solving engineering problems. Additionally, it will improve the ability to perform mathematical computations of the learned concepts using MATLAB.

**Module Outcomes: -** After Completion of this module, student will be able to

- Illustrate the knowledge of fundamental concepts of Partial differential equations.
- Apply suitable techniques to solve given engineering and scientific problems related to Partial differential equations based on the acquired knowledge.
- Analyze mathematical solutions of engineering and scientific problems related to Multivariable calculus, Partial differential equations.

Imagine you're baking a delicious pizza, and you are monitoring the temperature at a single point on your pizza (say, the center) over time

**Which differentiation you use here? ODE or PDE?**

If the temperature of the pizza changes at a specific point over time without considering spatial temperature variations across the pizza, you would employ an Ordinary Differential Equation (ODE) to model this scenario

Now, let's make it a bit more advanced! Instead of just looking at the middle, what if we want to see how hot or cold the whole pizza is? The temperature changes depending on both the time and where you're checking on the pizza

To capture this spatial variation, we need a first-order Partial Differential Equation (PDE). Here, temperature (T) is a function of both time (t) and position (x, y).

Rate of heat movement in the sideways and up-down directions: This relies on how easily heat moves through the pizza dough (thermal diffusivity - how fast heat spreads) and how temperature changes along the sides $(T_x)$and top-bottom $(T_y)$of the pizza. Heat movement helps to spread the temperature throughout the pizza.

But what about the movement of heat across the entire pizza? That's where the second-order PDE comes in. It goes beyond "how fast" by explaining how and why the temperature changes:

- It considers not just the temperature at the center, but also the temperatures at its neighboring points (like edges closer to the heating element).
- It accounts for how easily heat can move between these points depending on the pizza's ingredients and thickness.

The equation captures how these two things work together to change the temperature over time. it's basically a rule that says:

"The change in temperature at any point depends on how much hotter or cooler its neighbours are, and how quickly heat can move between them."

Why Second-Order Matters. The second-order PDE would include the second set of changes in temperature concerning $x$ , $(T_{xx})$ and y, $(T_{yy})$. These parts show how fast the heat spreads inside the pizza

Temperature (T): This represents the actual temperature at a specific point within the pizza dough.

**Position (x and y):** These variables indicate the location within the pizza dough where the temperature is being measured. Imagine a grid laid over your pizza; x and y would correspond to the coordinates of a particular point on that grid.

$\boldsymbol{T_x}$: This is the partial derivative of T with respect to x. It tells you how quickly the temperature changes as you move a small distance in the x-direction (across the pizza) at a specific point (considering y remains constant).

$\boldsymbol{T_y}$: This is the partial derivative of T with respect to y. It tells you how quickly the temperature changes as you move a small distance in the y-direction (up and down the pizza) at a specific point (considering x remains constant).

$\boldsymbol{T_{xx}}$: This is the partial derivative of $T_x$ with respect to x [This is the second-order partial derivative of T with respect to x]. It essentially captures the rate of change of the temperature gradient in the x-direction. It tells you how quickly the change in temperature ($T_x$) is varying across the pizza in the x-direction.

$\boldsymbol{T_{yy}}$**:** This is the partial derivative of $T_y$ with respect to y **[This is the second-order partial derivative of T with respect to y]**. It represents the rate of change of the temperature gradient in the y-direction. It tells you how quickly the change in temperature ($T_y$) is varying across the pizza in the y-direction.

The PDE (Partial Differential Equation) that models the temperature distribution in the pizza dough can be expressed,

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

- $T(x, y, t)$ is the temperature at a position (x,y) in a time t.
- α is the thermal diffusivity constant, which characterizes how fast heat diffuses through the pizza dough.

$$\frac{\partial^2 T}{\partial x^2} = ? \qquad \frac{\partial^2 T}{\partial y^2} = ?$$

**Why are $\boldsymbol{T_{xx}}$ and $\boldsymbol{T_{yy}}$ important in the second-order PDE?**

The first-order PDE with $T_x$ and $T_y$ describes the general flow of heat, While the second-order PDE with $T_{xx}$ and $T_{yy}$ gives a more detailed description of how quickly that heat flow is changing at different spots within the pizza,

## General Form of Second order PDE

We know that a partial differential equation is an equation that contains partial derivatives. In contrast to ODE's, where the unknown function depends only on one variables, in PDEs, the unknown function depends on several variables (like temperature $u(x,t)$ depends both on location $x$ and time $t$)

For the notational simplicity we use

$$u_t = \frac{\partial u}{\partial t};\, u_x = \frac{\partial u}{\partial x};\, u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

The unknown function $u$ always depends on more than one variable. The variable $u$ is called dependent variables, whereas the ones w differentiate with *respect to* are called independent variables.

Ex: $u_t = u_{xx}$

That the dependent variables $u(x,t)$ is a function of two independent variables $x$ and $t$, whereas in the equation $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

$u(r,\theta,t)$ depends on $r,\theta$ and t.

The order of PDE is the order of highest partial derivative in the equation.

Partial differential equations are either linear or nonlinear. In the linear ones, the dependent variable u and all its derivatives appear in a linear fashion (they are not multiplied together or squared, for ex- ample). More precisely, a second-order linear equation in two variables is an equation of the form

The general linear partial differential equation of the second order on two independent variable is of the form,

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + Fu = G$$

(Or)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + Fu = G \qquad \text{(i)}$$

Where A,B,C,D,E,F and G can be constants or given functions of $x$ and $y$. If $G = 0$ then, (i) is called Homogeneous otherwise it is called nonhomogeneous.

This is classified into

1. **Elliptic PDE's**: If $B^2 - 4AC < 0$, the PDE is elliptic. These PDEs are characterized by their smooth and continuous solutions. They are often associated with problems involving steady-state or equilibrium conditions.
   Properties:
   - Solutions exhibit no sudden changes or discontinuities.

- Boundary conditions are specified on the entire boundary of the domain.
- They are used to model problems such as heat conduction, electrostatics, and steady-state fluid flow.

2. **Parabolic PDE's**: If $B^2 - 4AC = 0$ the PDE is parabolic. PDEs involve both time and space variables and are characterized by their unique initial or boundary conditions.
   Properties:
   - Solutions evolve from an initial condition over time.
   - They are often associated with problems involving heat diffusion, time-dependent processes, or diffusion-reaction equations.
   - Boundary conditions are specified at one end of the domain or at initial time.

3. **Hyperbolic PDE's:** If $B^2 - 4AC > 0$the PDE is hyperbolic. In this case, the solutions exhibit wave-like behavior, with disturbances propagating along characteristic curves.
   Properties:
   - Solutions exhibit wave-like behavior and can develop shocks or discontinuities.
   - They are often associated with problems involving wave propagation, such as sound waves, electromagnetic waves, or fluid flow with shocks.
   - Boundary conditions are specified along characteristic lines or surfaces.

## Problems:

Classify the following equations:

1) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$

Ans: Comparing the equation with (1) above, we find that,

$$A = 1; B = 4; C = 4$$

$\therefore B^2 - 4AC = 4^2 - (4 * 1 * 4)$=0

So the equation is Parabolic.

**<u>Some of the Well Knows PDE's</u>**

- $\boldsymbol{u_t = u_{xx}}$ **(**Heat equation in one dimension)
- $\boldsymbol{u_t = u_{xx} + u_{yy}}$ (heat equation in two dimensions)
- $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (Laplace's Equation in polar form)
- $\boldsymbol{u_{tt} = u_{xx}}$ (one dimensional wave Equation)

## Practice problems:

Examine the following second-order PDE, Classify it as elliptic, parabolic, or hyperbolic, providing a brief justification based on its coefficients

1) $x^2\frac{\partial^2 u}{\partial x^2} + (1 - y^2)\frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty;\ -1 < y < 1$

Ans Ellipse , Parabola if $x = 0$

2) $(1+x^2)\frac{\partial^2 u}{\partial x^2} + (5+2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4+x^2)\frac{\partial^2 u}{\partial t^2} = 0$

Ans: $B^2 - 4AC = 9 > 0$ Hyperbolic

3) In which parts of the (x,y) plane is the following equation ?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x\, \partial y} + (x^2 + 4y^2)\frac{\partial^2 u}{\partial y^2} = 2sin(xy)$$

Ans: Outside the Ellipse, $\frac{x^2}{(0.5)^2} + \frac{y^2}{(0.25)^2} = 1$

4) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0.$

5) For a second-order linear PDE of the form $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$

Where A,B,C,D, E and F are constants, explain how you would determine its classification. Use the discriminant $B^2 - 4AC$ in your explanation.

6) Which type of second-order PDE represents steady-state phenomena?
7) What kind of physical processes does a hyperbolic second-order PDE typically model?
8) What kind of physical processes does a parabolic second-order PDE typically model?

## Finite Difference Approximation

Finite difference approximation is a numerical method used to approximate solutions to differential equations, particularly partial differential equations (PDEs). It works by dividing the problem's area into a grid and estimating the changes in the unknown function at each point on the grid.

Differential equations are powerful tools for modeling physical phenomena but analytical solutions are often difficult or impossible to find, especially for complex problems. Finite difference approximation provides a practical way to obtain numerical solutions to these equations by discretizing(refer to the process of dividing a continuous domain or variable into discrete intervals or points) the domain of the problem and approximating the derivatives of the unknown function at discrete points.

At the heart of finite difference approximation is the concept of discretization. Instead of considering the function and its derivatives as continuous functions, we divide the domain of the problem into a grid of discrete points in space and time. This allows us to approximate the derivatives of the function at each grid point using finite difference formulas.

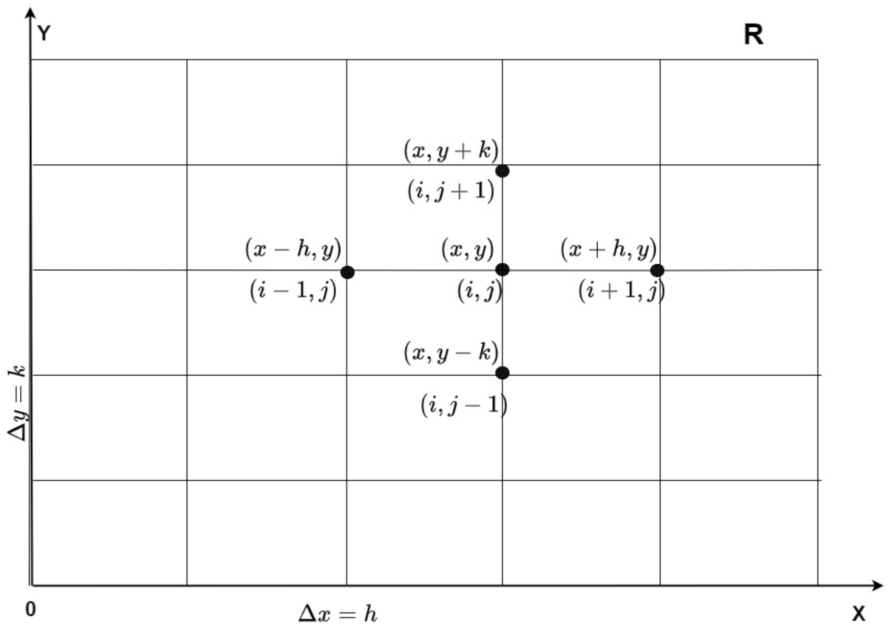
## Difference Operators:

Finite difference formulas are used to approximate the derivatives of the unknown function at each grid point. There are several types of finite difference operators, including forward, backward, and central differences. These operators approximate the derivative at a point using values of the function at neighboring points in the grid.

## Boundary Conditions:

Boundary conditions are essential constraints that must be applied at the edges of the grid to ensure that the numerical solution remains consistent with the physical problem being modeled.

Consider a rectangular region $R$ in the $x - y$ plane. Divide the region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in below fig. The points of intersection of dividing lines are called **mesh points, nodal points** or **grid points.**




Then we have the finite difference approximation for the partial derivatives in $x -$ direction is

$$u_x(x, y) = \frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h}$$

$$u_{xx}(x, y) = \frac{\partial^2 u}{\partial x^2} = \frac{u(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2}$$

Similarly we have the approximation for derivatives w.r.t t y:

$$u_y(x, y) = \frac{\partial u}{\partial y} = \frac{u(x, y+k) - u(x, y)}{k}$$

$$u_{yy}(x, y) = \frac{\partial^2 u}{\partial y^2} = \frac{u(x, y-k) - 2u(x, y) + u(x, y+k)}{k^2}$$

Above equations can be written in terms of $i$ and $j$ as follows,

$$u_x(x, y) = \frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

$$u_{xx}(x, y) = \frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$u_y(x, y) = \frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$u_{yy}(x,y) = \frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

## Solution of one-dimensional heat equation using Schmidt method:

Consider One dimensional Heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Where $c^2 = \frac{\lambda}{s\rho}$ (Where $\boldsymbol{s}$ is the specific heat of the material, λ is the thermal conductivity, $\boldsymbol{\rho}$ is the density)

**Thermal Conductivity $\boldsymbol{\lambda}$:**

- Thermal conductivity $\boldsymbol{\lambda}$ is a measure of a material's ability to conduct heat. It indicates how easily heat can pass through the material.

**Specific Heat $\boldsymbol{s}$ :**

- Specific heat $\boldsymbol{s}$ is the amount of heat required to raise the temperature of a unit mass of the substance by one degree Celsius.

**Density $\boldsymbol{\rho}$ :**

- Density $\rho$ is the mass per unit volume of the material.
- It represents how much mass is contained in each volume of the substance.

In the one-dimensional heat equation, these parameters are combined to form the thermal diffusivity $c^2 = \frac{\lambda}{s\rho}$ , which characterizes how quickly heat diffuses through the material. The values of these parameters depend on the specific properties of the material under consideration.

The solution of this equation is temperature function u(x, t) which is defined for values of x from 0 to $l$ and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions.

Consider a rectangular mesh in the $x - t$ plane with spacing $h$ along $x$ direction and $k$ along time $t$ direction. Denoting a mesh point $(x, t) = (ih, jk)$ ,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

And

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Substituting in Heat equation, we obtain

$$\boldsymbol{u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j}}$$

Where $\alpha = \frac{kc^2}{h^2}$ is the mesh ratio parameter,

Which is called Schmidt Explicit formula, and which is valid only for $0 \le \alpha \le \frac{1}{2}$.

Note: Schmidt method is the relation between the function values and two time levels $j$ and $j + 1$, so it is called two level formula.

If $\alpha = \frac{1}{2}$ then the above formula will be

$$\boldsymbol{u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)}$$

It is called Bendre-Schmidt Explicit formula.

**Example: Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$**

**Subject to the conditions $u(x,0) = \sin\pi x$, $0 \le x \le 1$; $u(0,t) = u(1,t) = 0$; Carry out computations for the two levels, taking $h = \frac{1}{3}$; $k = \frac{1}{36}$.**

Solution: Here $c^2 = 1, h = \frac{1}{3}, k = \frac{1}{36}$ so that $\alpha = \frac{kc^2}{h^2} = \frac{1}{4}$.

Also $u_{1,0} = sin\frac{pi}{3} = \frac{\sqrt{3}}{2}, u_{2,0} = sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and all other boundary values are zero as shown in fig.

We have the Schmidt formula,

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

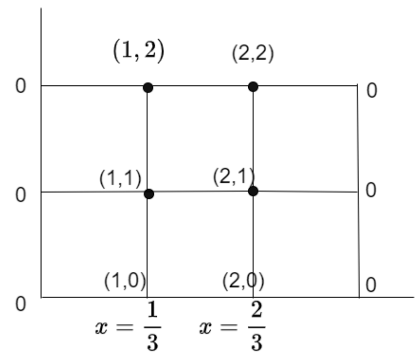
Becomes,

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + \alpha u_{i+1,j}\right]$$

For $j = 0$; $i = 1, 2$

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = \frac{1}{4}\left(0 + 2 * \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = 0.6495$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = \frac{1}{4}\left(\frac{\sqrt{3}}{2} + 2 * \frac{\sqrt{3}}{2} + 0\right) = 0.6495$$




For $j = 1$; $i = 1,2$

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.4871$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.4871$$

| j / i | 0 | 1 | 2 |
|---|---|---|---|
| **0** | **0** | **0** | **0** |
| **1** | $\frac{\sqrt{3}}{2}$ | **0.6495** | **0.4871** |
| **2** | $\frac{\sqrt{3}}{2}$ | **0.6495** | **0.4871** |
| **3** | **0** | **0** | **0** |

**Example: Determine the values of $u(x,t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the points $x = i$; $i = 0, 1, 2, 3, \ldots, 8$. And $t = \frac{1}{8}j$: $j = 0, 1, 2, \ldots, 5$.**

Solution: Here $c^2 = 4, h = 1$ and $k = \frac{1}{8}$. Then $\alpha = \frac{c^2k}{h^2} = \frac{1}{2}$

∴ we have Bendre Schmidt's formula,

$$\boldsymbol{u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})}$$

Now since $u(0,t) = 0 = u(8,t)$

∴ $u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of $j$, i.e., the entries in the first and last row are zero. Since

$$u(x,0) = 4x - \frac{1}{2}x^2$$

$$u_{i,0} = 4i - \frac{1}{2}i^2 = 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5;$$

For $i = 0,1,2,3,4,5,6,7$ $at$ $t = 0$

These are the entries of the first column.

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3.5 | 6.0 | 7.5 | 8.0 | 7.5 | 6.0 | 3.5 | 0 |
| 1 | 0 | 3.0 | 5.5 | 7.0 | 7.5 | 7.0 | 5.5 | 3.0 | 0 |
| 2 | 0 | 2.75 | 5.00 | 6.50 | 7.00 | 6.50 | 5.00 | 2.75 | 0 |
| 3 | 0 | 2.50 | 4.63 | 6.00 | 6.50 | 6.00 | 4.63 | 2.50 | 0 |
| 4 | 0 | 2.313 | 4.250 | 5.563 | 6.000 | 5.563 | 4.250 | 2.313 | 0 |
| 5 | 0 | 2.125 | 3.938 | 5.125 | 5.563 | 5.125 | 3.938 | 2.125 | 0 |

**Example:** Determine the temperature distribution of a long, thin rod with a length of 10 $cm$. with thermal conductivity $\lambda = 0.49\ cal/(s.cm.℃)$, $h = 2cm$, $k = 0.1\ S$. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 100℃$ and $T(10) = 50℃$. Note that the rod is aluminium with specific heat $s = 0.2174\ cal/g.℃$ and $\rho = 2.7g/cm^3$. Carry out the temperature distribution at all points up to t=0.2s.

Solution: Given Thermal conductivity $\boldsymbol{\lambda} = 0.49\ cal/(s.cm.℃)$, $h = 2\ cm$, $k = 0.1\ S$

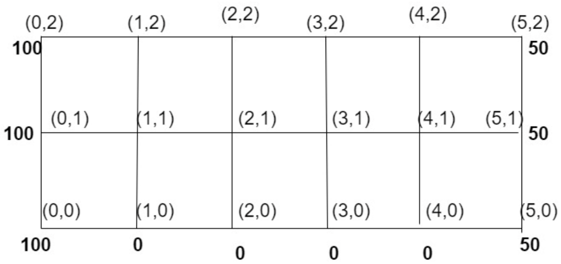
$c^2 = \frac{\lambda}{s\rho}$; $c^2 = \frac{0.49}{0.2174*2.7} = 0.8348\ cm^2/s$

$\alpha = \frac{kc^2}{h^2} = \frac{0.1*0.8348}{2^2}$=0.0209

When $t = 0$ the temperatures $[u_{0,0}, u_{0,1}, u_{0,2},]$ =0,

When $x = 0$ the temperatures $[u_{0,0}, u_{0,1}, u_{0,2}]$ =100; i.e., T(0)=100

When $x = 10$ the temperatures $[u_{5,0}, u_{5,1}, u_{5,2}]$ =50, ; i.e., T(10)=50




We have the Schmidt formula,

$$\boldsymbol{u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j}}$$

When i=1,j=0 (i.e at x=2 and t=0.1)

$$\boldsymbol{u_{1,1} = \alpha u_{0,0} + (1 - 2\alpha) u_{1,0} + \alpha u_{2,0}}$$

$$\boldsymbol{u_{1,1} = 0.0209(100 + (1 - 2 * 0.0209) * 0 + 0.0209 * 0 = 2.09}$$

Simillarly,

When $x = 2, 4, ,6 \; and \; 8$

$$u_{2,1} = 0; u_{3,1} = 0; u_{4,1} = 1.0438$$

When t=0.2s ,

$$u_{1,2} = 4.0878; u_{2,2} = 0.04352; u_{3,2} = 0.04788; u_{4,2} = 2.0439$$

| j<br>i | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 100 | 100 | 100 |
| 1 | 0 | 2.09 | 4.0878 |
| 2 | 0 | 1.0438 | 0.04352 |
| 3 | 0 | 1.0438 | 0.04788 |
| 4 | 0 | 1.0438 | 2.0439 |
| 5 | 50 | 50 | 50 |

Practice Problems:

1. A long, thin rod with a length of 10 $cm$ and the following values thermal conductivity $\lambda = 0.49 \; cal/(s.cm.℃)$, $h = 2cm$, $k = 0.1S$. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 100℃$ and $T(10) = 50℃$. Note that the rod is aluminium with specific heat $s = 0.2174 \; cal/g.℃$ and $\rho = 2.7g/cm^3$.
   a. Can you classify the above scenario into appropriate class of partial differential equations? Justify your answer. Write the PDE of the above question belongs to.

b. Identify the function involved in the given problem, clearly indicate the dependent and independent variables.
c. Give two applications of above class of PDE

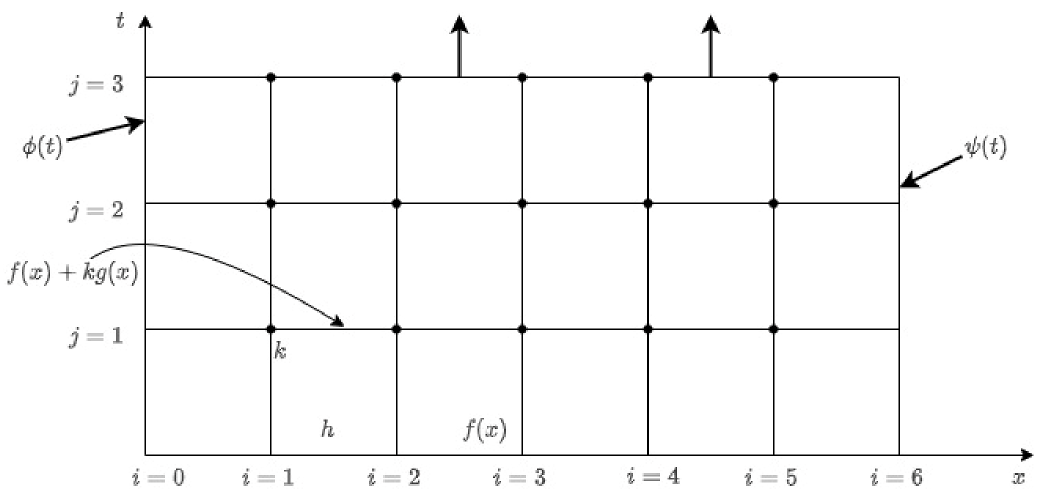
2. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ Subject to the conditions $u(x,0) = sin\pi x, 0 \le x \le 1; u(0,t) = u(1,t) = 0$; Carry out computations for the levels, taking $h = \frac{1}{3}; k = \frac{1}{36}$.
3. Determine the values of $u(x,t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 = u(5,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the points $x = i; i = 0,1,2,3, \dots ,5$. And $t = \frac{1}{8}j: j = 0,1,2, \dots ,5$.
4. Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = sin\pi x, 0 \le x \le 1$ using Schmidt method. Take h=0.2 and $\alpha = \frac{1}{2}$.
5. Solve the equation $2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ when $u(0,t) = u(4,t) = 0$ and $u(x,0) = x(4-x)$, taking $h = 1$ , $\alpha = \frac{1}{2}$ find the values up to t=5.
6. Solve the equation $\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = 0; f(0,t) = f(5,t) = 0, f(x,0) = x^2(25 - x^2)$; find the values of $f$ for $x = ih(i = 0,1,2, \dots ,5)$ and $t = jk(j = 0,1, \dots ,6)$ with $h = 1$ and $k = \frac{1}{2}$, using explicit method.
7. Determine the temperature distribution of a long, thin rod with a length of 10 $cm$ with thermal conductivity $\lambda = 0.49\ cal/(s.cm.℃)$, $h = 2cm$, $k = 0.1S$. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 100℃$ and $T(10) = 50℃$. Note that the rod is aluminium with specific heat $s = 0.2174\ cal/g.℃$ and $\rho = 2.7g/cm^3$. Carry out the temperature distribution at all points up to t=0.2s.
8. Determine the temperature distribution of a long, thin rod with a length of 15 $cm$ with thermal conductivity $\lambda = 0.58\ cal/(s.cm.℃)$, $h = 3cm$, $k = 0.15\ S$. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 120℃$ and $T(15) = 60℃$. Note that the rod is aluminium with specific heat $s = 0.0924\ cal/g.℃$ and $\rho = 8.9g/cm^3$. Carry out the temperature distribution at all points up to t=0.3s.
9. Determine the temperature distribution of a long, thin rod with a length of 12 $cm$ with thermal conductivity $\lambda = 0.52\ cal/(s.cm.℃)$, $\Delta x = 3cm$, $\Delta t = 0.2\ S$. At $t = 0$, the temperature of the rod is zero and the boundary conditions are fixed for all times at $T(0) = 90℃$ and $T(12) = 45℃$. Note that the rod is aluminium with specific heat $s = 0.093\ cal/g.℃$ and $\rho = 8.5g/cm^3$. Carry out the temperature distribution at all points up to t=0.6s.

## Solution of one-dimensional wave equation using explicit method:

The best example of hyperbolic partial differential equations is $u_{tt} = c^2 u_{xx}$.
Here $u_{tt}$ represents the acceleration of the wave. $u_{xx}$ represents how the wave changes spatially. c is the wave speed which indicates how fast the wave propagates through the medium.
Its solution is $u(x,t)$ represents the displacement of the wave at position $x$ and time $t$. defined for values of $x$ from 0 to $l$ and for $t$ from 0 to $\infty$, satisfying the initial and boundary conditions.
**Note:** In the case of hyperbolic equations we have two initial conditions and two boundary conditions.
Such kinds of equations arise from convective type of problems in vibrations, wave mechanics and gas dynamics.

## We seek the numerical solution of the wave equation

$u_{tt} = c^2 u_{xx}$-----(1)
Subject to the boundary conditions:
$u(0,t) = \varphi(t)$ -------(2)
$u(l,t) = \psi(t)$ -------(3)
And the initial conditions:
$u(x,0) = f(x)$ ------(4)
$u_t(x,0) = 0$ ---------(5)




Consider a rectangular mesh in the $x - t$ plane spacing $h$ along $x$ direction and $k$ along time $t$ direction. Denoting a mesh point $(x,t) = (ih, jk)$ as simply $i, j$ we have

$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$ and $u_{tt} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$

Substituting in wave equation we get

$$\therefore u_{i,j+1} = 2(1 - \alpha^2 c^2)u_{i,j} + \alpha^2 c^2 \left[u_{i-1,j} + u_{i+1,j}\right] - u_{i,j-1} \text{----(6)}$$

Where $\alpha = \frac{k}{h}$ is the mesh ratio

The coefficient of $u_{i,j}$ will vanish if $\alpha = \frac{1}{c}$ $or$ $k = \frac{h}{c}$ then equation (6) reduces to the form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \text{-----(7)}$$

This is called the ***explicit formula*** for the solution of the wave equation.

Further we express the initial condition (4) involving partial derivative w.r.t $t$ in terms of finite difference. We consider

$$u_t = \frac{1}{2k}\left[u_{i,j+1} - u_{i,j-1}\right]$$

Using equation (5) and taking $j = 0$ then we get $u_{i,1} = u_{i,-1}$ -----(8)

Using (8) and putting $j = 0$ then we get $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$ -----(9)

Further $u(0,t)$ and $u(l,t) = 0$ means $u_{0,j} = 0$ and $u_{l,j} = 0$ which implies that the values along the first column and last column are zero.

$u(x,0) = f(x)$ means $u_{i,0} = f(x)$ gives the values of $u$ along the first row.

Finally $u_t(x,0) = 0$ modified into the form (9) giving $u_{i,1}$ will give us the values of $u$ along the second row. These values will helps us to obtain the rest of the value of $u$ at the mesh points by the explicit formula (7). Thus we are able to determine $u(x,t)$ at all the interior mesh points.

**Note: 1.** *This provides an* **explicit scheme** *for the solution of the wave equation.*

**2.** *For $\alpha = 1/c$, the solution of (6) is stable and coincides with the solution of (1).*

**3.** *For $\alpha < 1/c$, the solution is stable but inaccurate.*

**4.** *For $\alpha > 1/c$, the solution is unstable.*

**5.** *The formula (6) converges for $\alpha \le 1$ i.e., $k \le h$.*

*6.* If $\alpha$ = 1/c, the solution of the explicit formula is stable and accurate because the time steps are precisely matched with the speed of wave propagation (c), allowing the numerical method to accurately capture the behavior described by the wave equation.

7. If $\alpha$ < 1/c, the solution remains stable because the time steps are small enough to prevent the numerical solution from blowing up. However, since the time steps are larger relative to the speed of wave propagation (c), the solution becomes less accurate. This is because the numerical method may not capture the rapid changes in the wave behavior as effectively due to the larger time steps.

8. If $\alpha$ > 1/c the solution becomes unstable because the time steps are greater than the wave propagation speed (c). As a result, there is instability and divergence in the numerical solution, producing inaccurate findings.

9. The explicit formula converges for $\alpha \le 1$ (k ≤ h), meaning that as long as the temporal step size (k) is less than or equal to the spatial step size (h), the numerical solution approaches the true solution of the wave equation as the step sizes approach zero.

**Example: Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$, taking $h = 1$ *upto* $t = 1.25$. The boundary conditions are $u(0,t) = u(5,t) = 0, u_t(x,0) = 0$and $u(x,0) = x^2(5-x)$.**

Solution: The wave equation in the standard form is $u_{tt} = c^2 u_{xx}$

Here $c^2 = 16$ or $c = 4$.

Since $h = 1$, we have $k = \frac{h}{c} = \frac{1}{4} = 0.25$

Now since $u(0,\ t) = u(5,\ t) = 0$, $\therefore\ u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j

i.e., the entries in the first and last rows are zero.

Since $u(x,0) = x^2(5-x)$

$\therefore\ u_{i,0} = i^2(5 - i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries for the first column.

Since $u_t(x, 0) = 0 \quad u_{i,1} = u_{i,-1}$,

We get $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 12) = 6$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4 + 18) = 11$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12 + 16) = 14$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the second column

Putting $j = 1$ in the explicit formula, we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the third column.

Similarly putting $j = 2, 3, 4$ successively in the explicit formula, the entries of the fourth, fifth, and sixth columns are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below:

| ***j*** **/ *i.*** | **0** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 4 | 6 | 7 | 2 | $-9$ | $-16$ |
| **2** | 12 | 11 | 8 | $-2$ | $-14$ | $-18$ |
| **3** | 18 | 14 | 2 | $-8$ | $-11$ | $-12$ |
| **4** | 16 | 9 | $-2$ | $-7$ | $-6$ | $-4$ |
| **5** | 0 | 0 | 0 | 0 | 0 | 0 |

**Example: The transverse displacement $u$ of a point at a distance $x$ from one end and at any time $t$ of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, with boundary conditions $u = 0$ at $x = 0, t > 0$ and $u = 0$ at $x = 4, t > 0$ and initial conditions $u = x(4-x)$ and $\frac{\partial u}{\partial t} = 0, 0 \le x \le 4$. Solve this equation numerically for one-half period of vibration, taking $h = 1$ and $k = 1/2$.**

Solution: Here, $\frac{h}{k} = 2 = c$.

∴ The difference equation for the given equation is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

which gives a convergent solution (since k < h).

Now since $u(0, t) = u(4, t) = 0$,

$u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of $j$.

i.e., the entries in the first and last rows are zero.

Since $u_{x,0} = x(4-x)$,

∴ $u_{i,0} = i(4-i) = 3, 4, 3$ for $i = 1, 2, 3$ at $t = 0$.

These are the entries of the *first column*.

Also $u_t(x, 0) = 0$ becomes,

$$\frac{1}{2k}\left[u_{i,j+1} - u_{i,j-1}\right] = 0$$

and taking $j = 0$ then we get $u_{i,1} = u_{i,-1}$

Putting $j = 0$ in the difference equation we obtain $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

Taking $i = 1, 2, 3$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = 2$$
$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3$$
$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 2$$

These are the entries of the second column.

Putting $j = 1$ in the explicit formula, we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0$$
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$$
$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$$

These are the entries of the third column and so on.

Now the equation of the vibrating string of length $l$ is $u_{tt} = c^2 u_{xx}$

∴ Its period of vibration $\frac{2l}{c} = \frac{2\times 4}{2} = 4sec$ $[\because l = 4 \text{ and } c = 2]$

This shows that we have to compute $u(x, 0)$ up to $t = 2$

i.e. Similarly we obtain the values of $u_{i,3}$ (fourth row) and $u_{i,4}$ (fifth row). Hence the values of $u_{i,j}$ are as shown in the next table:

| *j* / *i.* | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 |
| **1** | 3 | 2 | 0 | -2 | -3 |
| **2** | 4 | 3 | 0 | -3 | -4 |
| **3** | 3 | 2 | 0 | -2 | -3 |
| **4** | 0 | 0 | 0 | 0 | 0 |

**Practice Problems:**

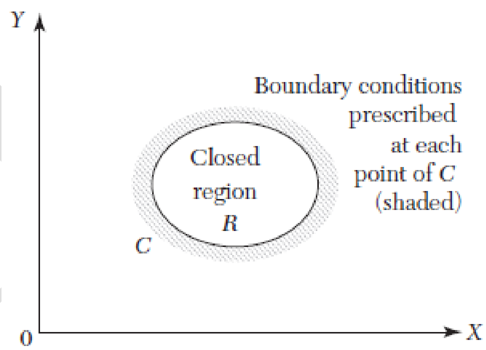
1. A string of length 8 meters is fixed at both ends. At time t = 0, the string is plucked such that it forms a triangular shape with maximum displacement at the midpoint. Provided for the range 0≤x≤8 and 0≤t≤2, with c=2 m/s, Δt=0.1 s, and Δx=0.5 m.
   a. Can you classify the above scenario into appropriate class of partial differential equations? Justify your answer.
   b. Identify the function involved in the given problem, clearly indicate the dependent and independent variables.
   c. Provide any two application of it.
2. Find the solution of the initial boundary value problem: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ , $0 \le x \le 1$ subject to the initial conditions $u(x,0) = sin\pi x, 0 \le x \le 1, \frac{\partial u}{\partial t}(x,0) = 0, 0 \le x \le 1$ and the boundary conditions $u(0, t) = 0$, $u(1, t) = 0$, $t > 0$; by using the explicit scheme by taking h=0.2.
3. Solve $y_{tt} = y_{xx}$ up to $t = 0.5$ with a spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x,0) = 0$ and $y(x, 0) = 10 + x(1 – x)$.
4. The transverse displacement of a point at a distance *x* from one end and at any time *t* of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}$ with the boundary conditions $u(0,t) = u(5, t) = 0$ and the initial conditions $u(x,0) = \begin{cases} 20x \; for \; 0 \le x < 1 \\ 5(5-x) \; for \; 1 \le x < 5 \end{cases}$ and $u_t(x,0) = 0$. Solve this equation numerically for one-half period of vibration, taking $h = 1$, $k = 0.2$.
5. . The transverse displacement $u$ of a point at a distance $x$ from one end and at any time $t$ of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, with boundary condition $u(0,t) = 0 = u(4,t)$

,t>0. for all $t$. Initial conditions $u = x(4- x)$ for $0 \le x \le 4$, and $\frac{\partial u}{\partial t}(x,0) = 0$. Using the explicit method, numerically Determine the transverse displacement $u(x,t)$ for $0 \le x \le 1$ and $0 \le t \le 0.5$ seconds y taking $h = 1$ and $k = \frac{1}{2}$.

6. The transverse displacement $u(x,t)$ of a guitar string of length $L$=1 meter satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ represents the displacement at position $x$ and time $t$. The string is fixed at both ends, i.e., $u(0,t) = u(1,t) = 0$ for all $t$. Initially, the string is plucked to a triangular shape: $u(x,0) = x(1- x)$ for $0 \le x \le 1$, and $\frac{\partial u}{\partial t}(x,0) = 0$. Using the explicit method, numerically solve for the transverse displacement $u(x,t)$ for $0 \le x \le 1$ and $0 \le t \le 0.5$ seconds.

## Elliptic Equations

The Laplace equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is an example of elliptic partial differential equations.
The Laplace equation describes how a scalar field varies in space, with no sources or sinks of the field and it arises in steady-state flow and potential problems. The solution of this equations is a function u(x, y) which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Figure).
In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

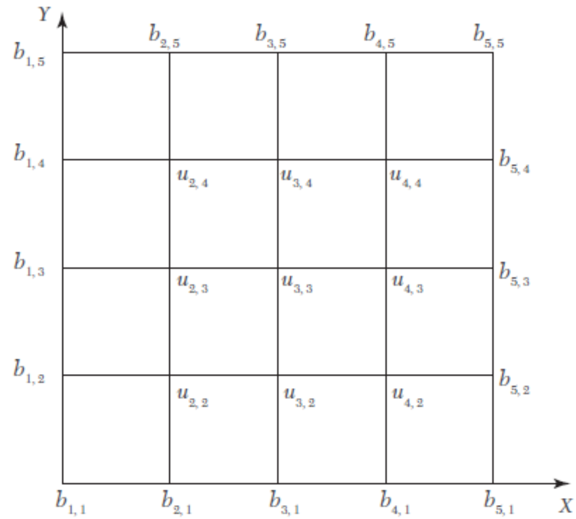



## Numerical solution of the Laplace's equation in two dimensions

Laplace's equation in two dimensions is:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{----(1)}$$

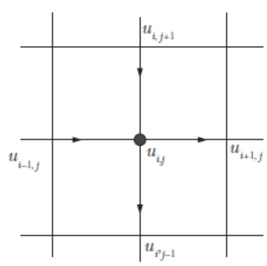
To solve Laplace's equation numerically using the finite difference method in a rectangular region R with known boundary conditions, we can discretize the domain into a grid of points and replace the derivatives in Laplace's equation with their finite difference approximations. Let's consider a rectangular region R divided into a network of square mesh with side length $h$ as shown in the figure




Replacing the derivatives in (1) by their difference approximations, we have

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right] \quad \text{------(2)}$$

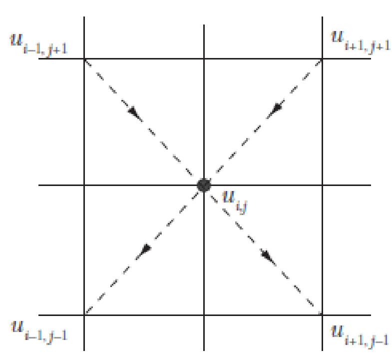
This is called the **standard five point formula**. This shows that the value of $u$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below which shown in the figure




Note: 1. Since the Laplace equation remains invariant when the co-ordinates are rotated through an angle of 45° we can also have the formula in the form

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right] \quad \text{------(3)}$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal five-point formula** which is represented in Figure



2. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

Now to find the initial values of u at the interior mesh points, we first use the diagonal five-point formula (3) and compute $u_{3,3}, u_{2,4}, u_{4,4}, u_{4,2}$ *and* $u_{2,2}$ in this order. Thus we get,

$$u_{3,3} = \frac{1}{4}\left(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}\right);$$
$$u_{2,4} = \frac{1}{4}\left(b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3}\right)$$
$$u_{4,4} = \frac{1}{4}\left(b_{3,5} + b_{5,3} + b_{3,5} + u_{3,3}\right);$$
$$u_{4,2} = \frac{1}{4}\left(u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3}\right)$$
$$u_{2,2} = \frac{1}{4}\left(b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1}\right)$$

The values at the remaining interior points, i.e., $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}\left(b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}\right),$$
$$u_{3,4} = \frac{1}{4}\left(u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3}\right)$$
$$u_{4,3} = \frac{1}{4}\left(u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}\right),$$
$$u_{3,2} = \frac{1}{4}\left(u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1}\right)$$

Having found all the nine values of $ui, j$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

In general,

$$u_{i,j}^{n+1} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n}\right]$$

This is called Liebmann's Process.

**Problems:**

**1. Solve the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the square mash of fig 1(a) with boundary values as shown.**

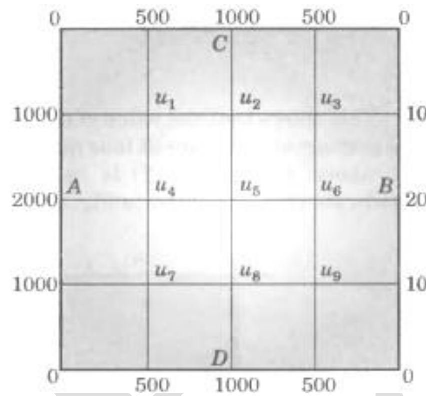
Solution. Let $u_1, u_2, \ldots, u_9$ be the values of $u$ at the interior mesh-points. Since the boundary values of $u$ are symmetrical about AB.

$\therefore \quad u_7 = u_1, u_8 = u_2, u_9 = u_3.$

Also the values of $u$ being symmetrical about $CD$, $u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values $u_1, u_2, u_4$ and $u_5$.

Now we find their initial value in the following order :




1(a)

$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$ (std. formula)

$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125$ (Diag. formula)

$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1188$ (std. formula)

$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1438$ (std. formula)

We carry out the iteration process using the formulae:

$$u_1^{(n+1)} = \frac{1}{4}\left(1000 + u_2{}^{(n)} + 500 + u_4{}^{(n)}\right)$$

$$u_2^{(n+1)} = \frac{1}{4}\left(u_1^{(n+1)} + u_1{}^{(n)} + 1000 + u_5{}^{(n)}\right)$$

$$u_4^{(n+1)} = \frac{1}{4}\left(2000 + u_5^{(n)} + u_1{}^{(n+1)} + u_1{}^{(n)}\right)$$

$$u_5^{(n+1)} = \frac{1}{4}\left(u_4^{(n+1)} + u_4^{(n)} + u_2{}^{(n+1)} + u_2{}^{(n)}\right)$$

First iteration: (put n=0)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) = 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1032 + 1000 + 1500) = 1141$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1032) = 1391$$

$$u_5^{(1)} = \frac{1}{4}(1391 + 1391 + 1141 + 1141) = 1266$$

Second iteration: (put n=1) $u_1^{(2)} = 1008, u_2^{(2)} = 1069, u_4^{(2)} = 1321, u_5^{(2)} = 1195$

Third iteration: $u_1^{(3)} = 973, u_2^{(3)} = 1035, u_4^{(3)} = 1288, u_5^{(3)} = 1162$

Fourth iteration: $u_1^{(4)} = 956, u_2^{(4)} = 1019, u_4^{(4)} = 1269, u_5^{(4)} = 1144$

Fifth iteration: $u_1^{(5)} = 947, u_2^{(5)} = 1010, u_4^{(5)} = 1260, u_5^{(5)} = 1135$

Sililarly, $u_1^{(6)} = 942, u_2^{(6)} = 1005, u_4^{(6)} = 1255, u_5^{(6)} = 1130$

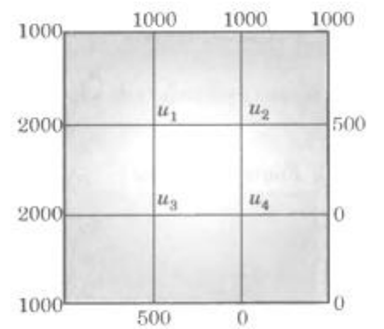
$u_1^{(7)} = 940, u_2^{(7)} = 1003, u_4^{(7)} = 1253, u_5^{(7)} = 1128$

$u_1^{(8)} = 939, u_2^{(8)} = 1002, u_4^{(8)} = 1252, u_5^{(8)} = 1127$

$u_1^{(9)} = 939,\ u_2^{(9)} = 1001, u_4^{(9)} = 1251\ ,\ u_5^{(9)} = 1126$
Hence $u_1 = 939,\ u_2^{\square} = 1001, u_4^{\square} = 1251\ ,\ u_5^{\square} = 1126.$

**2) Given the values of $u(x, y)$ on the boundary of the square in the fig. 2(a), evaluate the function $u(x, y)$ satisfying the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the pivotal points of this figure.**

Solution:To get the initial values of $u_1, u_2, u_3, u_4$ we assume $u_4 = 0.$ Then



2(a)

$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000$ (Diag. formula)
$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625$ (Std. formula)
$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875$ (Std. formula)
$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375$ (Std. formula)
We carry out the successive iterations, using the formulae

$$u_1^{(n+1)} = \frac{1}{4}\left(2000 + u_2{}^{(n)} + 1000 + u_3{}^{(n)}\right)$$
$$u_2^{(n+1)} = \frac{1}{4}\left(u_1^{(n+1)} + 500 + 1000 + u_4{}^{(n)}\right)$$
$$u_3^{(n+1)} = \frac{1}{4}\left(2000 + u_4^{(n)} + u_1{}^{(n+1)} + 500\right)$$
$$u_5^{(n+1)} = \frac{1}{4}\left(u_3^{(n+1)} + 0 + u_2{}^{(n+1)} + 0\right)$$

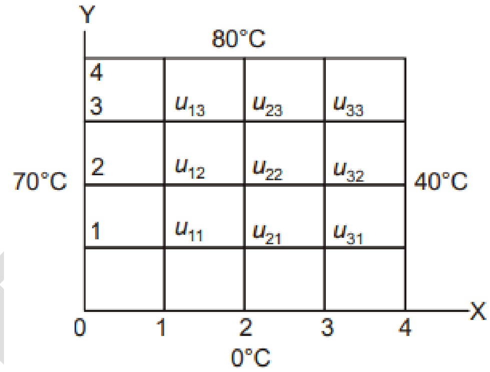
First iteration: (put n=0)

$$u_1^{(1)} = \frac{1}{4}(2000 + 625 + 1000 + 875) = 1125$$
$$u_2^{(1)} = \frac{1}{4}(1125 + 500 + 1000 + 375) = 750$$
$$u_3^{(1)} = \frac{1}{4}(2000 + 375 + 1125 + 500) = 1000$$
$$u_4^{(1)} = \frac{1}{4}(1000 + 0 + 750 + 0) = 438$$

Second iteration: (put n=1) $u_1^{(2)} = 1188,\ u_2^{(2)} = 782, u_3^{(2)} = 1032, u_4^{(2)} = 454$
Third iteration: $u_1^{(3)} = 1204, u_2^{(3)} = 789, u_3^{(3)} = 1040\ ,\ u_4^{(3)} = 458$
Fourth iteration: $u_1^{(4)} = 1207, u_2^{(4)} = 791, u_3^{(4)} = 1041\ ,\ u_4^{(4)} = 458$
Fifth iteration: $u_1^{(5)} = 1208, u_2^{(5)} = 791.5, u_3^{(5)} = 1041.5\ ,\ u_4^{(5)} = 458.25$
Hence $u_1 = 1208, u_2 = 792, u_3 = 1042, u_4 = 458.$

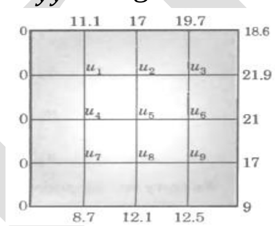
**3. Solve $u_{xx} + u_{yy} = 0$ for the temperature of the heated plate for the square region shown in figure below. Determine the temperature at the internal mesh points up to the third iteration . Give an estimate of the per cent error in the value of $u_{22}$.**

Solution: $u_{11}^{(1)} = 26.25, u_{21}^{(1)} = 9.84375, u_{31}^{(1)} = 10.691406,$
$u_{12}^{(1)} = 36.09375, u_{22}^{(1)} = 17.226562, u_{32}^{(1)} = 25.469238,$
$u_{13}^{(1)} = 69.785156, u_{23}^{(1)} = 62.629394, u_{33}^{(1)} = 78.036987$
Second iteration values: Error in $u_{22}^{(2)} = 65\%$
Third iteration : Error in $u_{22}^{(3)} = 13\%$

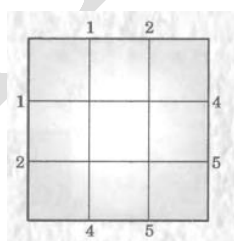



## Home work Problems:

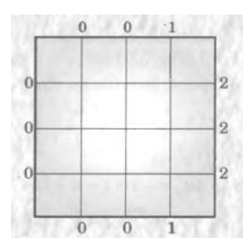
1. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that




2. Solve the equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. Iterate until the maximum difference between the successive values at any point in less than 0.001.




3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in fig. Iterate until maximum difference between successive values at any point is less than 0.005.



4. Solve $\nabla^2 u = 0$ under the conditions (h=1,k=1), $u(0,y) = 0, u(4,y) = 12 + y \; for \; 0 \leq y \leq 4; u(x,0) = 3x, u(x,4) = x^2 \; for \; 0 \leq x \leq 4$.
5. Consider a two-dimensional region with dimensions 4 cm by 4 cm. The Laplace equation for the temperature distribution, $\nabla^2 T = 0$, describes the steady-state heat conduction in the region.

Assume the following boundary conditions:

$$u(0,y) = 0, \; 0 \leq y \leq 4$$
$$u(4,y) = 12 + y, \; 0 \leq y \leq 4$$
$$u(x,0) = 3x, \; 0 \leq x \leq 4$$
$$u(x,4) = x^2, \; 0 \leq y \leq 4$$ carryout 2 iterations.

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