MODULE – 3
MULTIPLE INTEGRALS

CONTENTS:

- Evaluation of double and triple integrals
- Evaluation of double integrals by change of order of integration
- Evaluation of double integrals by changing into polar coordinates.
- Applications to find Area, Volume, and Total mass by double integral.

RBT LEVEL: L1, L2, L3

LAB COMPONENT:

Compute Area, Volume and Total Mass by multiple integration using MATLAB inbuilt functions.

LEARNING OBJECTIVES:

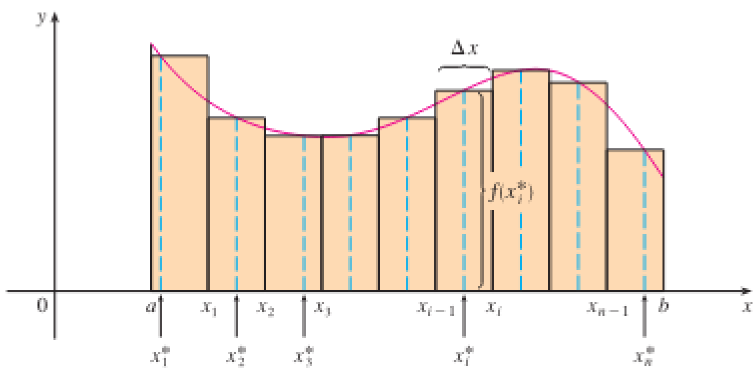
- Provide a comprehensive understanding of multiple integrals by extending the concept of single-variable integration to higher dimensions along with exploring their engineering applications.
- Enhance ability to perform mathematical computations of the learned mathematical concepts using MATLAB.

INTRODUCTION:

In this module we extend the idea of a definite integral to double integral and triple integrals of function of two variables and three variables.

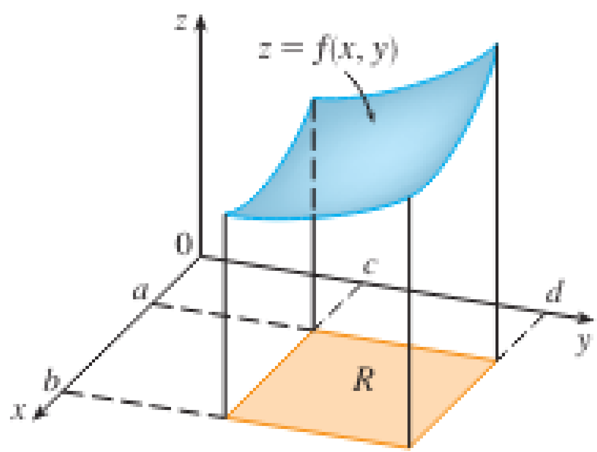
REVIEW OF THE DEFINITE INTEGRAL:

A definite integral for a function of one variable i.e., $\int_a^b f(x)dx$ is a number (area) where as an Indefinite integral $\int f(x)dx = F(x) + c$ is a function or family of functions.



DOUBLE INTEGRALS:

- In a definite integral we integrate a function $f(x)$ over an interval (a segment) of $x - axis$.
- In double integral we integrate a function $f(x, y)$ over a closed bounded region R in $xyplane$.



NOTE: R is the projection of $f(x, y)$ on to $xy - plane$

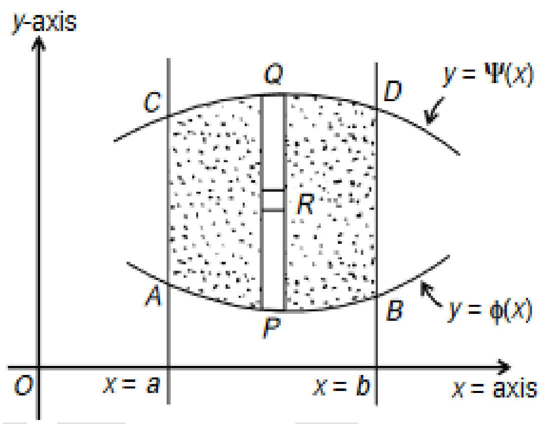
Double integral of a function $f(x,y)$ over a closed bounded region R is denoted as

$$\iint_R f(x,y)\ dx\,dy = \iint_R f(x,y)\ dA$$

Double integrals over a region R may be evaluated by two successive integrations as follows

TYPE I: When the region R is bounded by two continuous curves $y = \psi(x)$ and $y = \phi(x)$ and the two lines (ordinates) $x = a$ and $x = b$.

In such a case, integration is first performed with respect to y keeping x as a constant and then the resulting integral is integrated w.r.t x within the limits $x = a$ and $x = b$.



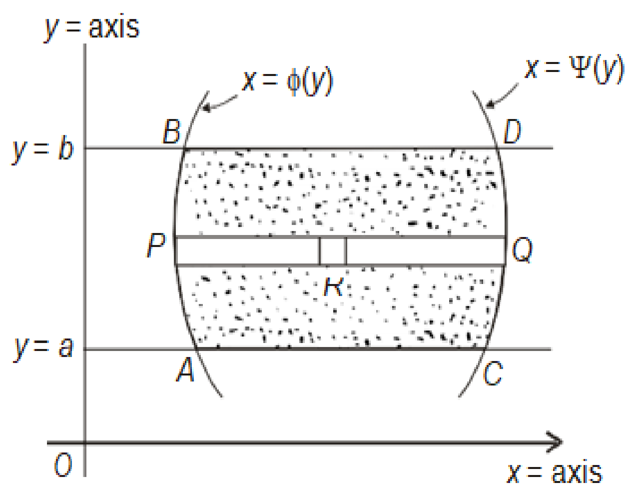
Mathematically expressed as:

$$\iint_R f(x,y)dxdy = \int_{x=a}^{x=b}\int_{y=\phi(x)}^{y=\psi(x)} f(x,y)dy\,dx$$

TYPE II: When the region R is bounded by two continuous curves $x = \phi(y)$ and $x = \psi(y)$ *and* the two lines (abscissa) $y = a$ *and* $y = b$.

In such a case, integration is first performed with respect to x. Keeping y as a constant and the resulting integral is integrated between the two limits $y = a$ *and* $y = b$.

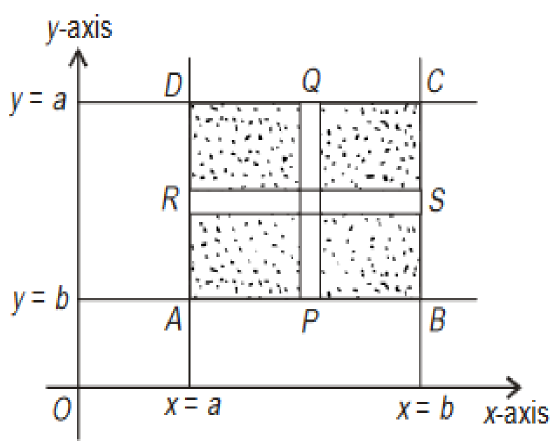
Fig. 2



Mathematically expressed as:

$$\iint_R f(x,y)dxdy = \int_{y=a}^{y=b} \int_{x=\phi(y)}^{x=\psi(y)} f(x,y)dy)\,dx$$

TYPE III: When both pairs of limits are constants, the region of Integration is the rectangle *ABCD* (say). In this case, it is immaterial whether f (x, y) is integrated first with respect to x or y, the result is unaltered in both the cases



This concept can be generalized to repeated integrals with three or more variable also.

TRIPLE INTEGRATION:

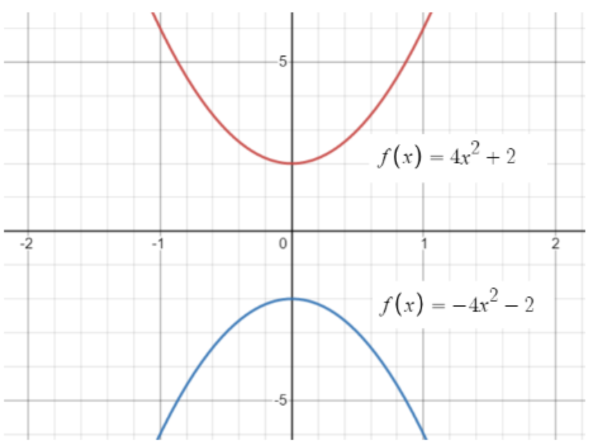
The triple integral of $f(x,y,z)$ over the region R is denoted by

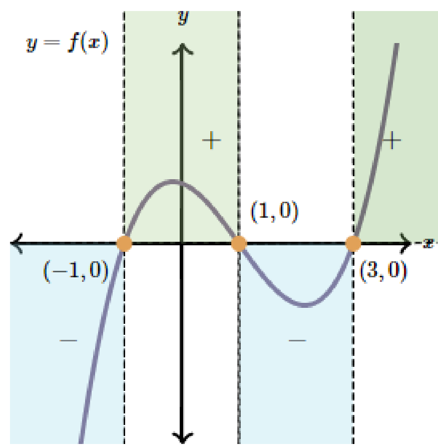
$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(x,y,z)\,dv$$

Triple integrals can be evaluated by three successive integrations, similar to double integrals of two successive integrations.

IMPORTANT DEFINITIONS: Let D be the defined domain.

1. **Positive function:** A function $f(x)$ is said to be a **positive function** if $\boldsymbol{f(x) > 0}$, **for all** $\boldsymbol{x \epsilon}$**D**
2. **Negative function:** A function $f(x)$ is said to be a **negative function** if $\boldsymbol{f(x) < 0}$, **for all** $\boldsymbol{x \epsilon}$**D**





3. **Even function:** A function $f(x)$ is said to be an even function if $f(-x) = f(x)$
4. **Odd function:** A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

Property: $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ f(x) is\ even \\ 0 & if\ f(x) is\ odd \end{cases}$

EVALUATION OF DOUBLE & TRIPLE INTEGRALS

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xydydx$ (Grewal-281)

Solution: Let I = $\int_0^1 \int_x^{\sqrt{x}} xydydx$

$= \int_0^1 [x.\frac{y^2}{2}]_x^{\sqrt{x}} dx$

$= \frac{1}{2}\int_0^1 x[x - x^2]dx$

$= \frac{1}{2}\int_0^1 [x^2 - x^3]dx$

$= \frac{1}{2}[\frac{x^3}{3} - \frac{x^4}{4}]_0^1$

$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{4}\right]$

$= \frac{1}{24}$

2. Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)dydxdz$ (Grewal-284)

Solution: Let $I = \int_{z=-1}^1 \int_{x=0}^z \int_{y=x-z}^{x+z} (x + y + z)dydxdz$

$= \int_{z=-1}^1 \int_{x=0}^z [yx + \frac{y^2}{2} + zy]_{x-z}^{x+z} dxdz$

$= \int_{z=-1}^1 \int_{x=0}^z [x(x + z - x + z) + \frac{1}{2}((x + z)^2 - (x - z)^2) + z(x + z - x + z)]\, dxdz$

$= \int_{z=-1}^1 \int_{x=0}^z [2xz + 2z^2 + 2xz]\, dxdz$

$= \int_{z=-1}^1 \int_{x=0}^z [4xz + 2z^2]dxdz$

$= \int_{z=-1}^1 [4z\frac{x^2}{2} + 2z^2x]_0^z dz$

$$= \int_{z=-1}^{1} [4z\left(\frac{z^2}{2} - 0\right) + 2z^2(z-0)]\, dz$$

$$= \int_{z=-1}^{1} 4z^3 dz$$

$$= z^4]_{-1}^{1}$$

$$= 1^4 - (-1)^4$$

$$= 0$$

Exercise Problems:

1. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)dxdy$ **(Grewal-275)**
   Soln: I =18880.2 nearly
2. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyzdzdydx$ **(Grewal-284)**
   Soln: I =1/48

Practice Problems:

1. Evaluate $\int_0^{\pi} \int_0^{a(1-cos\theta)} rsin\theta\, drd\theta$ **(Grewal-280)**

   Soln: $I = \frac{4a^2}{3}$
2. Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2)dzdydx$ **(Grewal-284)**
   Soln: $I = \frac{8abc}{3}(a^2+b^2+c^2)$

Homework Problems:

1. Prove that $\int_1^2 \int_3^4 (e^y + xy)\ dydx = \int_3^4 \int_1^2 (e^y + xy)\ dxdy$ **(Grewal-280)**
2. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx$ **(Grewal-284)**
   Soln: $I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$

EVALUATION OF DOUBLE INTEGRALS BY CHANGING THE ORDER OF INTEGRATION

Changing the order of integration: $dx\, dy \rightarrow dy\, dx$ or $dy\, dx \rightarrow dx\, dy$

i.e., $\int_{x=a}^{x=b}\int_{y=\emptyset(x)}^{y=\psi(x)} f(x,y)dy\, dx \;=\int_{y=c}^{y=d}\int_{x=g(y)}^{x=h(y)} f(x,y)dx\, dy$

The concept of change of order of integration evolved to help in handling typical integrals occurring in evaluation of double integrals.

In double Integral, the region of integration may be described more conveniently in one order than another. Changing the order can transform a difficult region into a simpler one, thereby simplifying the bounds of integration.

Thus, changing the order of integration can simplify the integral, making it easier to solve.

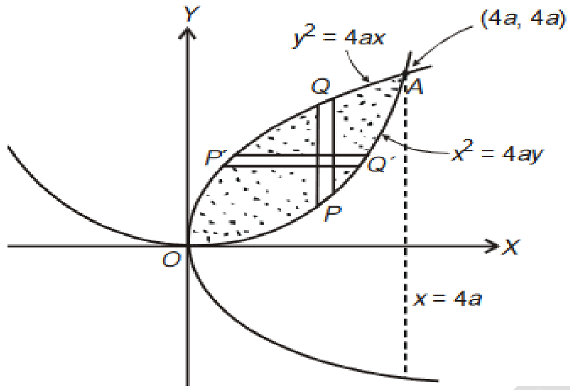
Some problems may be difficult to solve using TYPE I then we change to TYPE II and solve or VISE VERSA.

1. **Evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$ by changing the order of integration. (Grewal-277)**

Soln: Let $I = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$

Note that, the region of integration, R is bounded by the curves

- $y = \frac{x^2}{4a}$(ie $x^2 = 4ay$, a parabola symmetrical about y axis)
- $y = 2\sqrt{ax}$ (ie $y^2 = 4ax$, a parabola symmetrical about x axis)
- $x = 0$ (ie y axis)
- $x = 4a$ (ie a straight line parallel to y axis)



To find the points of intersection of the two parabolas:

$$\frac{x^2}{4a} = 2\sqrt{ax}$$

$$\Longrightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Longrightarrow x^4 = 64a^3x$$

$$\Longrightarrow x^4 - 64a^3x = 0$$

$$\Longrightarrow x(x^3 - 64a^3) = 0$$

$$\Longrightarrow x = 0 \ or \ x^3 - 64a^3 = 0$$

$$\Longrightarrow x = 0, x = 4a$$

$$\Longrightarrow y = 0, y = 4a$$

Therefore, the points of intersection are $(0,0)\&(4a, 4a)$

On changing the order of integration, we have

$$I = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dxdy$$

$$= \int_{y=0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= [2\sqrt{a}\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12a}]_0^{4a}$$

$$= 2\sqrt{a}\frac{(4a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(4a)^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Exercise Problems:

1. Evaluate $\int\int_R xydxdy$, where R is the region bounded by $y = x^2$ and $x + y = 2, 0 < x < 1$. **(Grewal-278)**

   Soln: $I = \int_0^1 \int_{x^2}^{2-x} xydydx = \frac{3}{8}$

2. Change the order of integration and hence evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx$ **(Grewal-281)**

   Soln: $I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy = 1$

Practice Problems:

1. Change the order of integration in $\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}$ and hence evaluate the same **(Grewal-280)**

   Soln: $I = \int_0^a \int_0^x \frac{xdydx}{x^2+y^2} = \frac{\pi a}{4}$

Homework Problems:

1. Evaluate $\int_0^1 \int_{e^x}^e \frac{dydx}{logy}$ by changing the order of integration **(Grewal-277)**

Soln:$I = \int_1^e \int_0^{logy} \frac{dxdy}{logy} = e - 1$

2. Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xydxdy$ and hence evaluate the same **(Grewal-281)**

Soln: $I = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xydydx = \frac{3}{2}a^4$

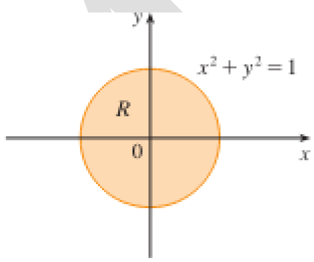
EVALUATION OF DOUBLE INTEGRALS BY CHANGING INTO POLAR COORDINTES

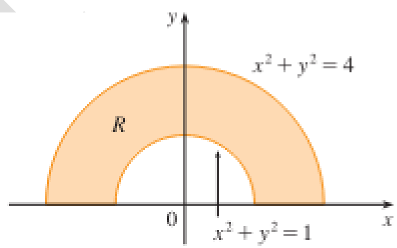
When the region of integration has boundaries that are more naturally described as arcs of circles, annuli or sectors, converting to polar form can make setting up the limits of integration easier.

To convert cartesian coordinates (x, y) to polar coordinates (r, θ):

We have $x = rcos\theta, y = rsin\theta$,

$$\int\int_R f(x,y)dxdy = \int\int_R f(r,\theta)rdrd\theta$$





1. **Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)dxdy$ by changing into polar coordinates. (Grewal-291)**

Solution: Let $I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} (x^2 + y^2)dxdy$

The region of integration is bounded by

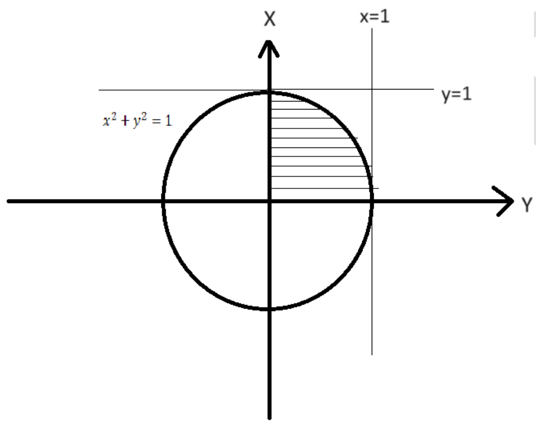
- $y = 0$ (x axis)

- $y = 1$ (a line parallel to x axis)
- $x = 0$ (y axis)
- $x = \sqrt{1-y^2}$ ($x^2 + y^2 = 1$, a unit circle centered at the origin)

Converting Cartesian coordinates to polar coordinates using

$x = rcos\theta, y = rsin\theta\ x^2 + y^2 = r^2\ and\ dxdy = rdrd\theta$

$$I = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2.rdrd\theta$$



$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^3 drd\theta$$

$$= \int_{r=0}^{1} r^3[\theta]_0^{\frac{\pi}{2}} dr$$

$$= \frac{\pi}{2}\int_{r=0}^{1} r^3 dr$$

$$= \frac{\pi}{2}[\frac{r^4}{4}]_0^1$$

$$= \frac{\pi}{2} \times \frac{1}{4}$$

$$= \frac{\pi}{8}.$$

Exercise Problems:

1. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$ by changing into polar co-ordinates. **(NP Bali M-6.30)**
   Soln: $I=\frac{\pi}{4}$
2. Evaluate $\int\int sin\pi(x^2+y^2)dxdy$ over the region bounded by the circle $x^2+y^2=1$by changing into polar coordinates. **(NP Bali M-6.34)**
   Soln: $I=2$

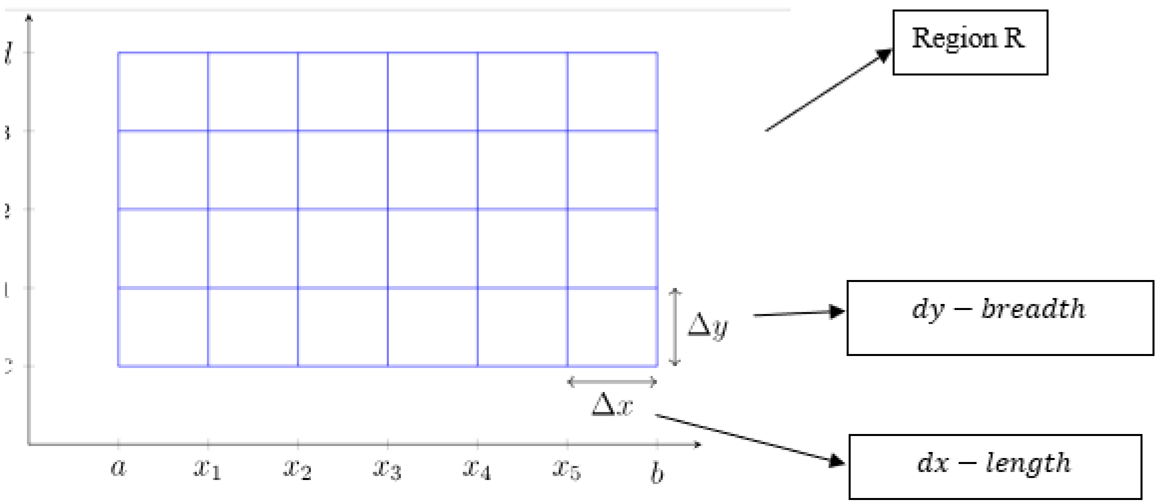
Practice Problems:

1. Evaluate by changing to polar coordinates $\int_0^2\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}(3x+4y^2)dydx$ **(James Stewart 976)**
   Soln: $I=\frac{28+15\pi}{2}$(recheck)

Homework Problem:

1. Evaluate by changing to polar coordinates $\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} cos(x^2+y^2)dxdy$ **(James Stewart 978)**
   Soln:$I=\frac{\pi sin9}{2}$

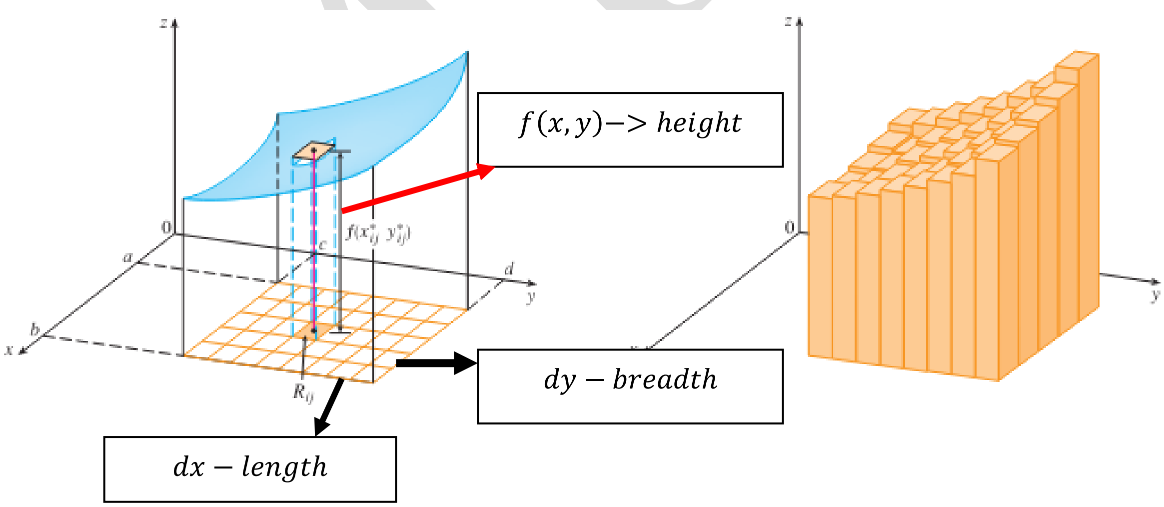
APPLICATION OF DOUBLE INTEGRAL: AREA, VOLUME AND TOTAL MASS



$Area = length * Breadth -->$

$Area\ of\ 1\ unit = dx * dy$

$Area\ A\ of\ Entire\ region\ R = \iint_R dx\, dy$



The **Volume V** beneath the surface $z = f(x, y)(> 0)$ and above a region R in the $xy - plane$ is

$$V = \iint_R f(x, y)\, dx\, dy$$

Let $f(x, y)$ be the density of a distribution of mass in the $xy - plane$. Then the **Total mass M** in R is

$$M = \iint_R f(x, y) dx\, dy$$

Exercise Problems:

1. Imagine you have a backyard space where you want to design a new garden. The plot of land has an irregular shape, defined by natural boundaries. Specifically, the lower boundary is the curve of $y = x^2$ and the upper boundary is a straight line at y=16. To maximize the use of this space for planting, estimate the exact area of this plot.
   Soln: 256/3
2. A former wants to cultivate vegetables by constructing a soil bed on a rectangular field of length (x) 2 units and breadth (y) 3 units. Estimate the amount of soil required to fill the Soil bed if the surface of the soil is given by $h(x, y) = x^2 + \frac{y}{2}$.
   Soln: 25/2
3. A rectangular flat roof, measuring 20 meters by 15 meters, has accumulated snow with a varying density due to recent heavy snowfall. The snow density at any point (x, y) on the roof is described by the function $\rho(x, y) = 2x + 3y\ g/m$ where x and y are the coordinates in meters from the bottom-left corner of the roof. Calculate the total mass of the snow on the roof.
   Soln: 12750g

Practice Problems:

1. A thin metal plate occupying the region R in the first quadrant bounded by the straight line passing through the origin which makes an angle 45^0 with the positive X axis and the parabola symmetrical about y axis which is open upwards whose vertex is at (0,0). The density of the plate at any point (x, y) is given by $\rho(x, y) = 4x + 2y$ grams per square centimeter. Calculate the total mass of the plate.

   Soln: $7(4a)^3/15$ grams.

2. A hobbyist maintains an aquarium with dimensions 6 feet length (x) and 4 feet width (y) and height is represented by the $f(x, y) = x + y$. He is curious to accommodate the right

number of fish without overcrowding. Assist the aquarist to estimate the amount of water that the aquarium can contain.

Soln: 120cubic feet

3. A forest in India has been severely affected by a forest fire. Satellite images show the burned area covering 10 kms horizontally has unique shape: the lower part forms a parabola $x^2 = 5y$, while the upper part forms a straight line $y = 2x$. Estimate the total area burnt due to the fire.

Soln: 33.3333 square kms