Module-4: PROBABILITY

CONTENTS

- Definition and examples,
- Algebra of Events,
- Addition theorem,
- conditional probability,
- Multiplication law,
- Baye's Theorem.

RBT LEVEL: L1, L2, L3

Learning objectives:

- Learning fundamentals of probability theory and its core concepts with examples.
- Apply set operations like unions and intersections to solve probability problems involving multiple events.
- Calculate probabilities of compound events using the addition theorem for both mutually exclusive and non-mutually exclusive events.
- Utilize conditional probability to compute event probabilities given specific conditions.
- Apply the multiplication law to solve complex probability problems involving multiple stages or conditions.
- Employ Bayes' Theorem to update probabilities based on new information across various disciplines.

APPLICATIONS:

1. **Weather Forecasting**: Meteorologists heavily rely on probability models to predict weather patterns. They use historical data and complex mathematical models to calculate the likelihood of certain weather conditions occurring in a specific area at a given time. These forecasts help people make informed decisions, such as whether to carry an umbrella or evacuate an area during a storm.

2. **Medical Diagnosis**: In medicine, probability is essential for diagnosing illnesses and determining the effectiveness of treatments. For instance, medical tests like mammograms or blood tests often produce results with a certain probability of accuracy. Doctors interpret these probabilities to make informed decisions about a patient's health and treatment plan.

3. **Insurance and Risk Management:** Insurance companies use probability to assess risks and set premiums. They analyze data on factors such as age, health history, and driving record to determine the likelihood of an event, such as illness, accident, or property damage, occurring. This information helps them calculate the appropriate premiums to charge customers.

4. **Financial Markets:** Investors and financial analysts use probability to assess investment opportunities and manage risk. Techniques like Monte Carlo simulations are used to model the range of possible

outcomes for investment portfolios based on different market scenarios and probabilities. This information helps investors make decisions about asset allocation and risk management.

5. **Manufacturing and Quality Control:** In manufacturing, probability is used to ensure product quality and reliability. Quality control processes involve sampling products from a production line and using statistical methods to assess the probability of defects or failures. This information helps manufacturers identify and address issues before products reach consumers.

6. **Game Theory:** Probability is central to game theory, which is used to analyze strategic interactions and decision-making in various fields, including economics, politics, and biology. For example, in competitive markets, firms use probability models to predict their competitors' actions and optimize their strategies accordingly.

7. **Traffic Engineering:** Traffic engineers use probability models to analyze traffic flow patterns, predict congestion, and optimize road networks. By understanding the probabilities of different traffic scenarios, they can design more efficient transportation systems and implement strategies like traffic signal timing and lane management.

INTRODUCTION

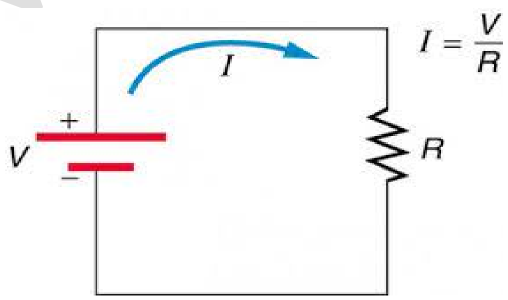
"Whenever we use mathematics in order to study some observational phenomena, we must essentially begin by building a mathematical model (deterministic or probabilistic) for these phenomena".

Experiments done in physics/ chemistry/ electronics ?

Input —How it's related ?→ output

For a specific or defined input, we get a deterministic output.

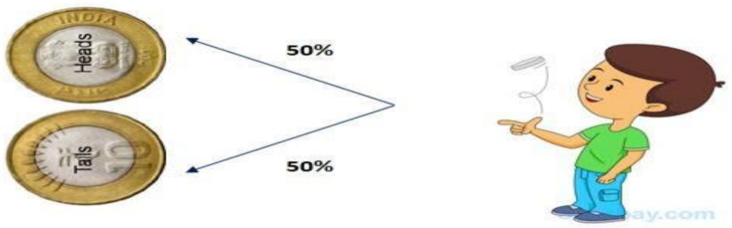
Example-1:



- For a particular value of V and R, we can **determine** the current $I = \frac{V}{R}$ through the circuit using Ohm's Law.

- There might be small variations in the value of a current due to several factors but still we will get the appropriate values.
- We get deterministic output for different input or same input.

Example-2:



- If we toss a coin, we **cannot determine** with certainty that we will get head or tail.
- Even if the same person tossed the same coin under similar conditions, still we cannot predict whether we will get head or tail.

If 2 persons do the physics experiment individually in lab, do they get same or near values as an output?

If the same 2 persons do the experiment of tossing of a coin 3 to 4 times, do they get same output?

What have you observed in Example 1 and Example 2?

Example-1 is a **deterministic model** and **Example-2** is **probabilistic** (Non deterministic) **model.**

- In a deterministic model it is supposed that the actual outcome (whether numerical or otherwise) is determined from the conditions under which the experiment or procedure is carried out.
- In a non- deterministic(probabilistic) model, however, the conditions of experimentation determine only the probabilistic behavior (more specifically, the probabilistic law) of the observable outcome.

Let us look at the following scenarios

E1: Toss a die and observe the number that shows on top.
E2: Toss a coin four times and observe the sequence of heads and tails obtained.
E3: Manufacture items on a production line and count the number of defective items produced during 24 hours.
E4: An airplane wing is assembled with a large number of rivets. The number of defective rivets is counted.
E5: A light bulb is manufactured. It is then tested for its life length by inserting it into a socket and the time elapsed (in hours) until it burns out is recorded.

E6: A lot of 10 items contain 3 defectives. One item is chosen after another (without replacing the chosen item) until the last defective item is obtained. The total number of items removed from the lot is counted.

What do the above experiments have in common?

The following features are pertinent for our characterization of a **random experiment.**

Probability deals with unpredictability and randomness, and **probability theory** is the branch of mathematics that is concerned with the study of random phenomena. A random phenomenon is one that, under repeated observation, yields different outcomes that are not deterministically predictable.

Definitions:

1. **Experiment:** An **experiment** is any process of trial and observation.
2. **Random experiment**: An experiment whose outcome is uncertain before it is performed is called a random experiment.
3. **Outcome:** The result of a random experiment is called the **outcome**.
4. **Trial**: Performing a random experiment is called a **trial**.
5. **Event:** An **event A** (for a particular sample space S associated with an experiment) is a set of possible outcomes. In set terminology, an event is a subset of the sample space S.

Example: A concerned authority wants to study whether an accident will happen or not at a particular junction.

- This is a **random experiment** as we cannot predict whether an accident will happen or not.
- Let's assume an accident happened. Then that will be the **outcome** of the experiment**.**
- We are performing this experiment each day which is nothing but **trials.**
- Possible outcomes are - happening of an accident, not happening of an accident which are nothing but the **events.**

What is the difference between an Outcome and an Event?

Let's toss 2 coins and assume that we get HH which will be the Outcome of that experiment.

But which are the set of all possibilities for this experiment? HH, HT, TH, TT

So, each of these HH, HT, TH, TT we call it as events of this experiment.

6. **Sample space:** The collection of possible elementary outcomes is called the **sample space** of an experiment, which is usually denoted by **S**.
   Example: If a coin is tossed twice, then the sample space is given by

$$S = \{HH, HT, TH, TT\}$$

7. **Size of the sample space** is denoted by $\boldsymbol{n(S)}$.

   If a coin is tossed twice, exhaustive number of cases is 2. (i.e., HH, HT, TH, TT).

8. **Favourable Cases or events:** The number of outcomes of a random experiment which results in the happening of an event are termed as the cases favourable to the event.

   Example: In a toss of two coins, the number of cases favourable to the event "*Exactly one head*" is 2 (i.e., HT, TH) and for getting "*two heads*" is 1(i.e., HH)

9. **Equally likely cases:** The outcomes are said to be equally likely or equally probable if each of the event has the equal chance or possibility of happening.

   Example: If we roll a die, all the outcomes (the faces 1,2,3,4,5,6) are equally likely if the die is unbiased.

10. **Mutually Exclusive Events or Cases:** Two or more events are said to be mutually exclusive if the happening of any one of them excludes the happening of all others in the same experiment.

    Example: In tossing of a coin, the events "head" and "tail' are mutually exclusive because if head comes, we can't get tail. If tail comes, we can't get head.
    In set notation, we can say that intersection is empty (i.e., $A \cap B = \emptyset$)

11. **Independent Events:** Events are said to be independent of each other if happening of any one of them not affected by and does not affect the happening of any one of others.
    In set notation, we can say that $P(A \cap B) = P(A) \cdot P(B)$

    Example: In tossing of a coin repeatedly, the event of getting 'tail' in 1st throw is independent of getting "tail" in second, third or subsequent throws.

Classical definition of Probability:

The probability $P(A)$ of an event A is the ratio of the number of outcomes N_A of an experiment that are favourable to A to the total number N of possible outcomes of the experiment. That is,

$$\boldsymbol{P(A) = \frac{N_A}{N}}$$

Axiomatic definition of Probability:

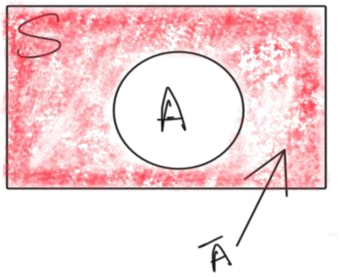
Consider a random experiment whose sample space is S. For each event A of S we assume that a number $P(A)$, called the probability of event A, is defined such that the following hold:

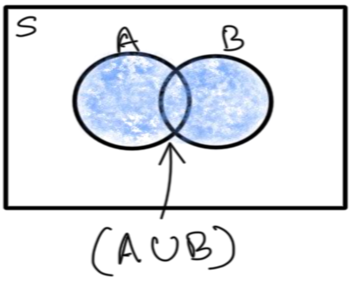
Axiom 1: $0 \le P(A) \le 1$, which means that the probability of A is some number between and including 0 and 1.

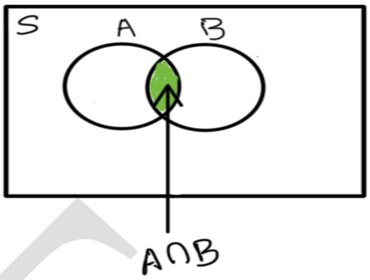
Axiom 2: $P(S) = 1$, which states that with probability 1, the outcome will be a sample point in the sample space.

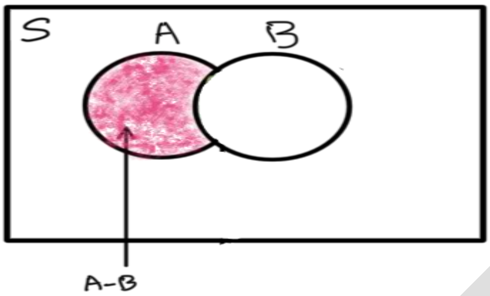
Axiom 3: For any set of n mutually exclusive events $A_1, A_2, \dots, A_n$ defined on the same sample space, $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

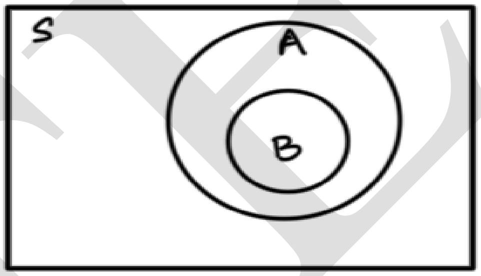
Venn Diagram:











$\boldsymbol{B \subset A}$

Properties of Probability

1. $P(\bar{A}) = 1 - P(A)$, which states that the probability of the complement of A is one minus the probability of A.
2. $P(\emptyset) = 0$, which states that the impossible (or null) event has probability zero.
3. If $A \subset B$, then $P(A) \le P(B)$. That is, if A is a subset of B, the probability of A is at most the probability of B (or the probability of A cannot exceed the probability of B).
4. $P(A) \le 1$, which means that the probability of an event A is at most 1.
5. If $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then
$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
6. $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Other identities include the following:

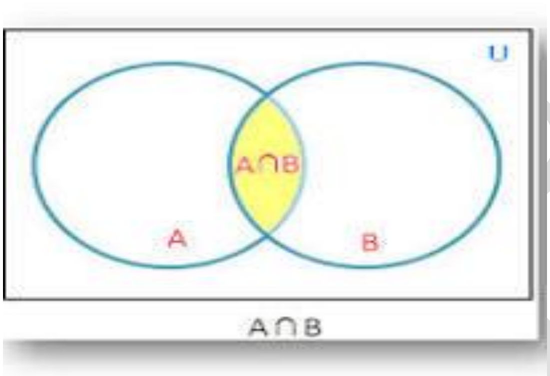
- $A - B = A \cap \bar{B}$, which states that the difference of A and B is equal to the intersection of A and the complement of B.
- $A \cup S = S$, which states that the union of A and the universal set S is equal to S.
- $A \cap S = A$, which states that the intersection of A and the universal set S is equal to A.
- $A \cup \emptyset = A$, which states that the union of A and the null set is equal to A.
- $A \cap \emptyset = \emptyset$, which states that the intersection of A and the null set is equal to the null set.
- $\bar{S} = \emptyset$, which states that the complement of the universal set is equal to the null set.
- For any two sets A and B, $A = (A \cap B) \cup (A \cap \bar{B})$, which states that the set A is equal to the union of the intersection of A and B and the intersection of A and the complement of B.
- **De Morgan's Law:**
  1. $\overline{\boldsymbol{A} \cup \boldsymbol{B}} = \bar{\boldsymbol{A}} \cap \bar{\boldsymbol{B}}$
  2. $\overline{\boldsymbol{A} \cap \boldsymbol{B}} = \bar{\boldsymbol{A}} \cup \bar{\boldsymbol{B}}$

- Since $A \cap B \subseteq A \subseteq A \cup B$, we have
  $\boldsymbol{P(A \cap B) \leq P(A) \leq P(A \cup B)}$

ADDITION THEOREM

If A and B are any two events, then $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

why we have to subtract $\boldsymbol{P(A \cap B)}$?



When we consider $P(A)$, we have already considered $P(A \cap B)$. But, we do $P(A) + P(B)$ then the intersection part is added twice. Hence, we subtract the $P(A \cap B)$ at the end once.

For any 3 events A, B, C we can prove that

$$\boldsymbol{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)}$$

Example: $\boldsymbol{P(A) = 0.3, P(B) = 0.4, P(A \cap B) = 0.2}$, find $\boldsymbol{P(A \cup B)}$.
Solution: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.2 = 0.5$

Example: A card is drawn from a well-shuffled pack of playing cards. Find the probability that it is either a spade or an ace?

Solution: The probability that the card drawn is either a spade or an ace is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$

Example: Krishna is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, Find the probability that he will get at least one offer from these two companies?

Solution: Using the additive rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9$$

Practice Problems:

1. $P(\bar{A}) = 0.6, P(\bar{B}) = 0.7$ and $P(A \cup B) = 0.1$. Find $P(A \cap B)$.
   Solution: 0.6

2. In a certain computer center, 47% of the programmers can program in FORTRAN, 35% in MATLAB, and 20% in PYTHON, and every programmer can program in at least one of these languages. If the probability that a randomly chosen programmer can program in FORTRAN and MATLAB is 0.23, PYTHON and FORTRAN are 0.12, and MATLAB and PYTHON are 0.11, determine the probability that a randomly chosen programmer can program in all three languages.
   Solution: 0.44

3. The probability that a contractor will get a plumbing contract is $\frac{2}{3}$, and the probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting at least one contract is $\frac{4}{5}$, what is the probability that he will get both the contracts?
   Solution: $\frac{14}{45}$

Homework Problems:

1. $P(A) = 0.3, P(B) = 0.7, P(A - B) = 0.2$, find $P(A \cup B)$?
   Solution: 0.9

2. A card is drawn at random from a pack of 52 cards. Find the probability that the drawn card is either club or a queen.
   Solution: $\frac{4}{13}$

3. In a game of Scrabble, you draw a tile randomly from the bag. What is the probability of drawing a vowel (a, e, i, o, u) or a consonant?
   Solution: 1

CONDITIONAL PROBABILITY

Consider the following experiment. We are interested in the sum of the numbers that appear when two dice are tossed. Suppose we are interested in the event that the sum of the two tosses is 7, and we observe that the first toss is 4. Based on this fact, the six possible and equally likely outcomes of the two tosses are {4, 1}, {4, 2}, {4, 3}, {4, 4}, {4, 5}, and {4, 6}. In the absence of the information that the first toss is 4, there would have been 36 sample points in the sample space. But with the information on the outcome of the first toss, there are now only 6 sample points. Let A denote the event that the sum of the two dice is 7, and let B denote the event that the first die is 4.

The conditional probability of event A given event B, denoted by P(A|B), is defined by

$$P(B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{4,3\})}{P(\{4,1\}) + P(\{4,2\}) + P(\{4,3\}) + P(\{4,4\}) + P(\{4,5\}) + P(\{4,6\})} = \frac{1/36}{6/36} = \frac{1}{6}$$

Note:

- $P(A|B)$ is only defined when $P(B) > 0$.
- The notion of conditional probability provides the capability of reevaluating the idea of probability of an event in light of additional information, that is, when it is known that another event has occurred.
- The probability $P(A|B)$ is an updating of $P(A)$ based on the knowledge that event B has occurred.

Example: A fair coin was tossed two times. Given that the first toss resulted in heads, find the probability that both tosses resulted in heads?

Solution: Since the coin is fair, the four sample points of the sample space $S = \{HH, HT, TH, TT\}$ are equally likely.
Let X denote the event that both tosses came up heads; i.e., $X = \{HH\}$
Let Y denote the event that the first toss came up heads; i.e., $Y = \{HH, HT\}$
The probability that both tosses resulted in heads, given that the first toss resulted in heads, is given by

$$P(X|Y) = \frac{P(X\cap Y)}{P(Y)} = \frac{1/4}{2/4} = \frac{1}{2}$$

Example: The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$, the probability that it arrives on time is $P(A) = 0.82$ and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a flight,

a) arrives on time, given that it departed on time.
b) departed on time, given that it has arrived on time.

Solution:

a) The probability that a flight arrives on time, given that it departed on time is

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

b) The probability that a flight departed on time, given that it has arrived on time is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

Practice Problems:

1. A box contains 4 bad and 6 good tubes. 2 tubes are drawn out from the box at a time. One of them is found to be good. Determine the probability that the other one is also good.
   Solution: $\frac{5}{9}$
2. If A and B be events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$. Find

a) $P(A|B)$
b) $P(B|A)$
c) $P(A \cap \bar{B})$
d) $P(A|\bar{B})$

Solution:

a) $\frac{1}{3}$, **b)** $\frac{1}{4}$, **c)** $\frac{1}{4}$, **d)** $\frac{1}{3}$

3. The probability that a management trainee will remain with a company is 0.60. The probability that an employee earns more than ₹ 10,000 per month is 0.50. The probability that an employee is a management trainee who remained with the company or who earns more than ₹ 10.000 per month is 0.70. What is the probability that an employee earns more than ₹ 10,000 per month, given that he is a management trainee who stayed with the company?
   Solution: $\frac{2}{3}$

4. A box of 100 gaskets contains 10 gaskets with type A defects. 5 gaskets with type B defects and 2 gaskets with both types of defects. Find the probabilities that,
   i. a gasket to be drawn has a type B defect under the condition that it has a type A defect.
   ii. a gasket to be drawn has no type B defect under the condition that it has no typed A defect.

   Solution:

   i. 0.2
   ii. 0.97

Homework Problems:

1. Suppose cards numbered from one to ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?
   Solution: $\frac{1}{6}$

2. A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space S is given by $S = \{(b,b),(b,g),(g,b),(g,g)\}$, and all outcomes are equally likely. [(b,g) means, for instance, that the older child is a boy and the younger child is a girl.]
   Solution: $\frac{1}{3}$

3. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?
   Solution: 0.08

THE PRODUCT RULE, OR THE MULTIPLICATIVE RULE:

Definition: If in an experiment the events A and B can both occur, then

$\boldsymbol{P(A \cap B) = P(A)\, P(B|A)}$, provided $\boldsymbol{P(A) > 0}$

Thus, the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability that B occurs, given that A occurs. Since the events $A \cap B$ and $B \cap A$ are equivalent, it follows from the above statement that we can also write

$$\boldsymbol{P(A \cap B) = P(B \cap A) = P(B)\, P(A|B)}$$

In other words, it does not matter which event is referred to as A and which event is referred to as B.

Example 1:

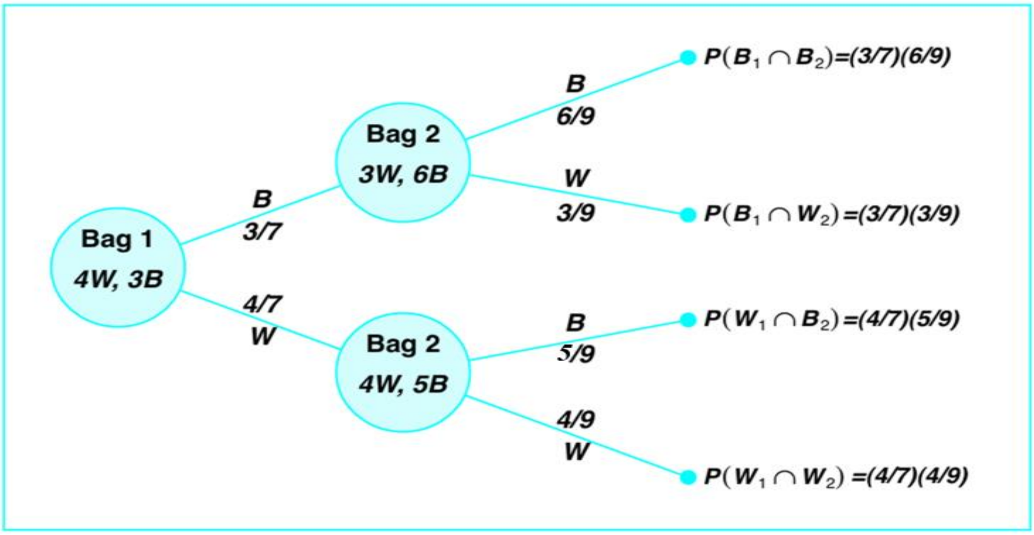
First bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution: Let B_1, B_2, and W_1 represent the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1 respectively.

We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in the Figure.

Now,

$$P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$
$$= P(B_1)\, P(B_2|B_1) + P(W_1)\, P(B_2|W_1)$$
$$= \left(\frac{3C_1}{7C_1}\right)\left(\frac{6C_1}{9C_1}\right) + \left(\frac{4C_1}{7C_1}\right)\left(\frac{5C_1}{9C_1}\right) = \frac{38}{63}$$



Example 2:

Suppose a 'cubical die' is thrown. The sample space $S = \{1, 2, 3, 4, 5, 6\}$.Let A be the event of getting an odd number and B be the event of getting a number less than 4. Find the probability of getting a number less than 4 being an odd number.

Solution: Let A be the event of getting an odd number. i.e., $A = \{1, 3, 5\}$

In the next trial let B be the event of getting a number less than 4. i.e., $B = \{1, 2, 3\}$

Now, $P(B|A)$ is the probability of getting a number less than 4 being an odd number.

$$\text{i.e., } P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Here, $A \cap B = \{1, 3\}$ and $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$, $P(A) = \frac{3}{6} = \frac{1}{2}$

$$\therefore P(B|A) = \frac{1/3}{1/2} = \frac{2}{3}$$

Remark:
Two events A and B are independent if and only if

$$\boldsymbol{P(A \cap B) = P(A)P(B)}$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Example 1:
A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Solution: Let A and B represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B) = (0.98)(0.92) = 0.9016$$

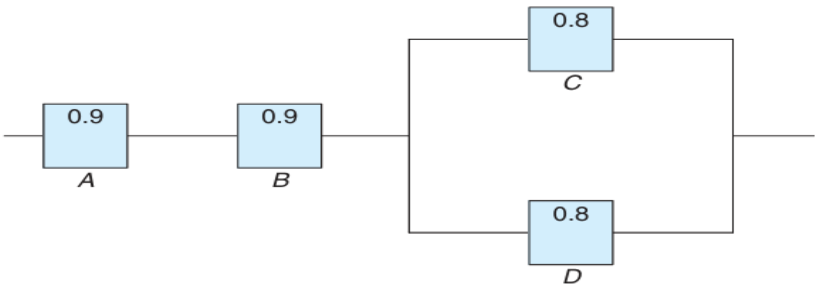
Example 2:
Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution: We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective, then we interpret $A \cap B$ as the event that A occurs and then B occurs after A has occurred. The probability of first removing a defective fuse is $P(A) = \frac{5C_1}{20C_1} = \frac{1}{4}$,

then the probability of removing a second defective fuse from the remaining 4 is
$P(B|A) = \frac{4C_1}{19C_1} = \frac{4}{19}$.
Hence, $P(A \cap B) = P(A)\,P(B|A) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19}$

Practice Problems:

1. **An electrical system consists of four components as illustrated in Figure below. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in Figure below. Find the probability that**
   i. **The entire system works**
   ii. **The component C does not work, given that the entire system works. Assume that the four components work independently.**



Solution: In this configuration of the system, A, B, and the subsystem C and D constitute a serial circuit system, whereas the subsystem C and D itself is a parallel circuit system.

i. Clearly the probability that the entire system works can be calculated as follows:

$$P[A \cap B \cap (C \cup D)] = P(A)P(B)P(C \cup D)$$
$$= P(A)P(B)[1 - P(\overline{C \cup D})]$$
$$= P(A)P(B)[1 - P(\bar{C} \cap \bar{D})]$$
$$= P(A)P(B)[1 - P(\bar{C})\,P(\bar{D})]$$
$$= (0.9)(0.9)[1 - (1 - 0.8)(1 - 0.8)] = 0.7776$$

The equalities above hold because of the independence among the four components.

ii. To calculate the conditional probability in this case, notice that

$$= \frac{P(A \cap B \cap \bar{C} \cap D)}{P(\textit{the system works})} = \frac{(0.9)(0.9)(1-0.8)(0.8)}{0.7776} = 0.1667$$

2. **A bag contains 15 black & 10 white balls. Two balls are drawn in succession. Find the probability that one of them is black & the other is white.**

Solution:

case 1: Let A be the event of drawing a black ball as the first & B be an event of drawing a white ball as the second.

$$P(A) = \frac{15}{25},\ \ P(B|A) = \frac{10}{24}$$

∴ Probability of drawing a black ball as first & white as second

$$= P(A \cap B) = P(A)\, P(B|A) = \left(\frac{15}{25}\right)\left(\frac{10}{24}\right) = \frac{1}{4}$$

case 2: Let C be the event of drawing a white ball as the first & D be an event of drawing a black ball as the second.

$$P(C) = \frac{10}{25},\ \ P(D|C) = \frac{15}{24}$$

∴ Probability of drawing a white ball as first & black as second

$$= P(C \cap D) = P(C)\, P(D|C) = \left(\frac{10}{25}\right)\left(\frac{15}{24}\right) = \frac{1}{4}$$

Thus, Probability $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

3. **There are two bags. Bag B_1 contains 2 silver coins and 4 copper coins, and bag B_2 contains 4 silver coins and 4 copper coins. If a coin is selected randomly from one of the two bags. What is the probability that it is a silver coin.**

Solution: Let E be an event of selecting a silver coin.

Case 1: Let B_1 be an event of selecting the first bag and the probability of getting a silver coin from the first bag denoted by $P(E|B_1)$

$$P(B_1) = \frac{1}{2}\ , P(E|B_1) = \frac{2C_1}{6C_1} = \frac{1}{3}$$

⇒ Probability of selecting the first bag & getting silver coin is,

$$P(B_1 \cap E) = P(B_1)\, P(E|B_1) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$$

Case 2: Let B_2 be an event of selecting the second bag and the probability of getting a silver coin from the second bag denoted by $P(E|B_2)$

$$P(B_2) = \frac{1}{2}\ , P(E|B_2) = \frac{4C_1}{8C_1} = \frac{1}{2}$$

⇒ Probability of selecting the second bag & getting silver coin is,

$$P(B_2 \cap E) = P(B_2)\, P(E|B_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

∴ The probability that it is a silver coin $= \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

4. **Two coins are tossed. Let A denote the event "at most one head on the two tosses," and let B denote the event "one head and one tail in both tosses." Verify whether A and B are independent events?**

Solution: The sample space of the experiment is $S = \{HH,\ HT,\ TH,\ TT\}$.
Now, events are defined as follows:

$A = \{HT, TH, TT\}$ and $B = \{HT, TH\}$.
Also, $A \cap B = \{HT, TH\}$

$P(A) = \frac{3}{4}, P(B) = \frac{2}{4} = \frac{1}{2}, \ P(A \cap B) = \frac{2}{4} = \frac{1}{2}, P(A)P(B) = \frac{3}{8}$

Since $P(A \cap B) \neq P(A)P(B)$, we conclude that events A and B are not independent.

Homework Problems:

1. Suppose a 'cubical die' is thrown. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event of getting an even number and B be the event of getting a number less than 6. Find the probability of getting a number less than 6 being even number.
   Solution: $\frac{2}{3}$

2. Box B_1 contains 5 blue & 6 green marbles, another box B_2 contains 6 blue & 4 green marbles. A box is selected randomly and a marble is drawn from it.
   a) What is the probability that the marble drawn is blue?
   b) Given that the marble drawn is blue. What is the probability that it came from box B_1?
   Solution: a) $\frac{29}{55}$, **b)** $\frac{25}{58}$

3. Suppose that we have a tool box containing 25 tools, out of which 10 are defective. If 2 tools are selected at random and removed from the box in succession without replacing the first. What is the probability that both tools are defective?
   Solution: $\frac{3}{20}$

BAYE'S THEOREM:

If the events $B_1, B_2, \dots, B_n$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for

$i = 1, 2, \dots, n$, then for any event A of S,

Consider the Venn diagram. Event A is seen to be the union of the mutually exclusive events.

i.e., $B_1 \cup B_2 \cup B_3 \cup B_4 \cup \dots \cup B_n = S$
then for any event A of S, $A = S \cap A$

Now, $A = (B_1 \cup B_2 \cup B_3 \cup B_4 \cup \dots \cup B_n) \cap A$

$$\text{i.e., } A = (B_1 \cap A) \cup (B_2 \cap A) \cup \dots \cup (B_n \cap A)$$

$$P(A) = P[(B_1 \cap A) \cup (B_2 \cap A) \cup \dots \cup (B_n \cap A)]$$

$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_n \cap A)$$

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i)\, P(A|B_i)$$

Note: $P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots + P(B_n)P(A|B_n)$ is defined as **the total probability** of an event A.

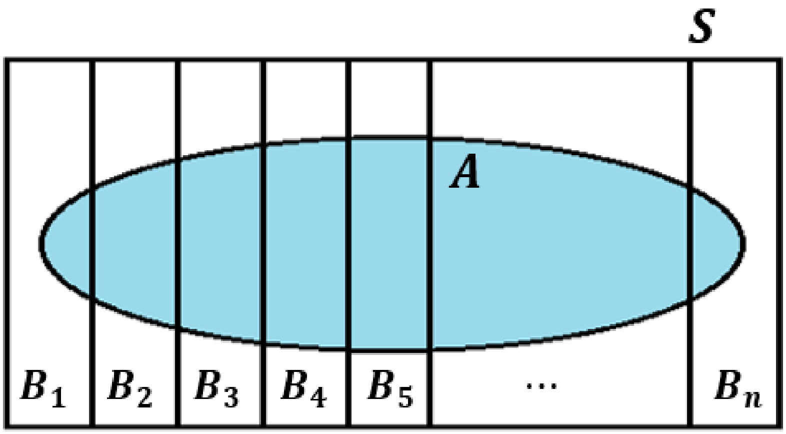


Figure: Partitioning the sample space S

If the events $B_1, B_2, \dots, B_n$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \dots, n$, then for any event A in S such that $P(A) \neq 0$.

Note: $B_i \cap B_j = \emptyset$, for any i and j, $i \neq j$.

By the definition of conditional probability,

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i \cap A)}{\sum_{i=1}^{n} P(B_i \cap A)}$$

$$\boldsymbol{P(B_i|A) = \frac{P(B_i)\, P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A|B_i)}}$$

Example 1:

In a manufacturing unit, three machines B_1, B_2, and B_3, make 30%, 45% and 25% of the products respectively. It is known from past experience that 2%, 3% and 2% of the products made by each machine are defective. Now, suppose that a finished product is randomly selected. Find the probability that it is defective. What is the probability that it was made by machine B_3?

Solution: Given B_1, B_2 and B_3 are three machines. Consider the following events:

B_1: the product is made by machine B_1 $\Rightarrow P(B_1) = \frac{30}{100} = 0.3$

B_2: the product is made by machine B_2 $\Rightarrow P(B_2) = \frac{45}{100} = 0.45$

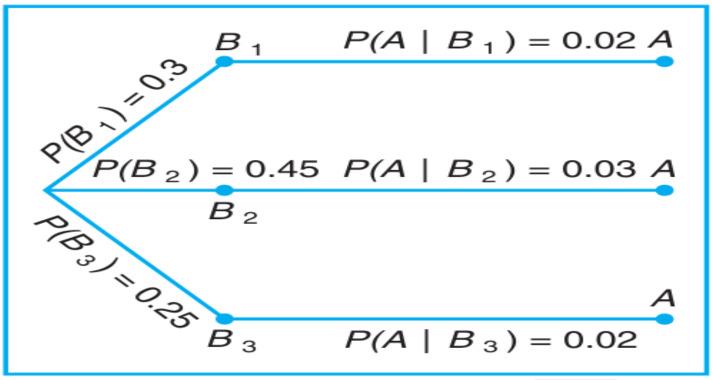
B_3: the product is made by machine B_3 $\Rightarrow P(B_3) = \frac{25}{100} = 0.25$

Let A be an event that a finished product selected at random is defective.

$$P(A|B_1) = \frac{2}{100} = 0.02$$

$$P(A|B_2) = \frac{3}{100} = 0.03$$

$$P(A|B_3) = \frac{2}{100} = 0.02$$



The probability of finished product selected at random is defective,

$$P(A) = \sum_{i=1}^{3} P(B_i)\, P(A|B_i) = P(B_1)\, P(A|B_1) + P(B_2)\, P(A|B_2) + P(B_3)\, P(A|B_3)$$

$$P(A) = (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02)$$

$$P(A) = 0.006 + 0.0135 + 0.005$$

$$P(A) = 0.0245$$

The probability of finished product selected at random is defective and it was made by machine B_3,

$$P(B_3|A) = \frac{P(B_3)\, P(A|B_3)}{P(A)} = \frac{0.005}{0.0245} = \frac{10}{49} = 0.2041$$

Example 2:

A student buys 1000 integrated circuits (ICs) from supplier A, 2000 ICs from supplier B, and 3000 ICs from supplier C. He tested the ICs and found that the conditional probability of an IC being defective depends on the supplier from whom it was bought. Specifically, given that an IC came from supplier A, the probability that it is defective is 0.05; given that an IC came from supplier B, the probability that it is defective is 0.10; and given that an IC came from supplier C, the probability that it is defective is 0.10. If the ICs from the three suppliers are mixed and one is selected at random, what is the probability that it is defective? Find the probability that it came from supplier A.

Solution: Let $P(A)$, $P(B)$, and $P(C)$ denote the probability that a randomly selected IC came from suppliers A, B, and C, respectively.

Also, let $P(D|A)$ denote the conditional probability that an IC is defective, given that it came from supplier A,

$P(D|B)$ denote the conditional probability that an IC is defective, given that it came from supplier B,

$P(D|C)$ denote the conditional probability that an IC is defective, given that it came from supplier C.

$$P(D|A) = 0.05$$

$$P(D|B) = 0.10$$

$$P(D|C) = 0.10$$

$$P(A) = \frac{1000}{1000 + 2000 + 3000} = \frac{1}{6}$$

$$P(B) = \frac{2000}{1000 + 2000 + 3000} = \frac{1}{3}$$

$$P(C) = \frac{3000}{1000 + 2000 + 3000} = \frac{1}{2}$$

Let $P(D)$ denote the probability that a randomly selected IC is defective. Then, from the principle of total probability,

$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$$

$$= (0.05)\left(\frac{1}{6}\right) + (0.10)\left(\frac{1}{3}\right) + (0.10)\left(\frac{1}{2}\right)$$

$$= 0.0917$$

The probability that the randomly selected IC came from supplier A, given that it is defective,

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{0.0083}{0.0917} = 0.0909$$

Example 3: The chance that a doctor will diagnose a disease correctly is 65%. The chance that a patient will die after correct diagnosis will be 40% and the chance of death by wrong diagnosis is 70%. If a patient dies, what is the probability that the disease was correctly diagnosed?

Solution: Let A be an event that a patient dies

Let B_1 be an event that a doctor will diagnose a disease correctly

Let B_2 be an event that a doctor will diagnose a disease incorrectly

$$\Rightarrow P(B_1) = \frac{65}{100} = 0.65$$

$$\Rightarrow P(B_2) = \frac{35}{100} = 0.35$$

Probability that a patient will die after correct diagnosis is $P(A|B_1) = \frac{40}{100} = 0.4$

Probability that a patient will die after wrong diagnosis is $P(A|B_2) = \frac{70}{100} = 0.7$

The probability that a patient dies is $P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$

$$P(A) = (0.65)(0.4) + (0.35)(0.7) = 0.505$$

The probability that the disease was correctly diagnosed, if a patient dies is,

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(A)} = \frac{(0.65)(0.4)}{0.505} = 0.5149$$

Example 4: Three major parties A, B & C are in power for the election in a state and the chance of their winning in the election is of the ratio $1:3:5$. These parties A, B & C have probability of banning the online lottery is $\frac{2}{3}, \frac{1}{3}$ & $\frac{3}{5}$ .What is the probability of banning the online lottery in the state. What is the probability that the ban is from party C.

Solution: Given, that there are three major parties A, B & C.

The probability that party A winning in the election is $P(A) = \frac{1}{9}$

The probability that party B winning in the election is $P(B) = \frac{3}{9}$

The probability that party C winning in the election is $P(C) = \frac{5}{9}$

Let E be an event of banning the online lottery

$\Rightarrow P(E|A) = \frac{2}{3}$

$\Rightarrow P(E|B) = \frac{1}{3}$

$\Rightarrow P(E|C) = \frac{3}{5}$

The probability of banning the online lottery in the state is,

$$P(E) = P(A)P(E|A) + P(B)P(E|B) + P(C)P(E|C)$$

$$P(E) = \left(\frac{1}{9}\right)\left(\frac{2}{3}\right) + \left(\frac{3}{9}\right)\left(\frac{1}{3}\right) + \left(\frac{5}{9}\right)\left(\frac{3}{5}\right) = \frac{14}{27} = 0.5185$$

The probability that the ban is from party C is,

$$P(C|E) = \frac{P(C)P(E|C)}{P(E)} = \frac{1/3}{0.5185} = 0.6429$$

Practice Problems:

1. An insurance company has insured 4000 doctors, 8000 teachers, and 12000 businessmen. The probabilities of a doctor, teacher, and businessman dying before the age of 58 are 0.01, 0.03, and 0.05, respectively. If one of the insured individuals dies before 58, find the probability that he is a doctor. **Solution: 0.0455**

2. In a certain neighborhood, 90% of children fell ill due to the flu and 10% due to measles, with no other diseases reported. The probability of observing rashes for measles is 0.95 and for the flu is 0.08. If a child develops rashes, find the probability of the child having the flu. **Solution: 0.4311**

3. In a bolt factory, there are four machines A, B, C & D, manufacturing $20\%, 15\%, 25\%$ and 40% of the total output respectively. It is known from past experience that $5\%, 4\%, 3\%$ and 2% of the bolts made by each machine are defective. A bolt is chosen at random & found to be defective. What is the probability that it was manufactured by machine A or D? **Solution: 0.5715**

Homework Problems:

1. In a bolt factory, three machines A, B & C, manufacturing $25\%, 35\%$ and 40% of the total output respectively. It is known that $5\%, 4\%$ and 2% are defective bolts. A bolt is chosen at random & found to be defective. What is the probability that it was manufactured by machine C?

   Solution: 0.2319

2. A bag contains 3 coins, one of which is 'two headed' coin and the other two coins are fair & unbiased. A coin is chosen at random from a bag and tossed 4 times in succession. If head turns up each time, what is the probability that this is two headed coin.

   Solution: 0.8889

3. Three boxes contain 1 white, 2 red, 3 green marbles; 2 white, 1 red, 1 green marbles; and 4 white, 5 red, 3 green marbles. Two marbles are drawn from a box at random, they are found to be 1 white & 1 green. Find the probability that the marbles drawn came from third box.

   Solution: 0.2542