Module-1

PARTIAL DIFFERENTIATION AND VECTOR DIFFERENTIATION

CONTENTS:

- Partial Derivatives.
- Total Derivatives.
- Maximum and Minimum Values.
- Scalar and Vector fields.
- Gradient of a scalar field, Directional Derivatives.
- Divergence of a vector field, Solenoidal vector.
- Curl of a vector field, Irrotational vector.
- Physical interpretation of Gradient, Divergence and Curl.

RBT LEVEL: L1, L2, L3, L4

LAB COMPONENTS:

- Solve optimization problems.
- Compute Gradient, Divergence, Curl and its Geometrical representation.

LEARNING OBJECTIVES:

- Impart knowledge of the derivative of multivariable functions and their applications in Engineering and vector differentiation to analyse physical phenomena involving magnitude and direction.
- Enhance the ability to perform Mathematical computations using MATLAB.

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The Meteorological Service of Canada has devised the humidex (or temperature- humidity index) to describe the combined effects of temperature and humidity.

The above case study explains that the humidex I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$.

Functions of two variables:

Just as the graph of a function $\boldsymbol{f}$ of one variable is a curve $\boldsymbol{C}$ with equation $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, so the graph of a function $\boldsymbol{f}$ of two variables is a surface $\boldsymbol{S}$ with equation $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$. We can visualize the graph $\boldsymbol{S}$ of as lying directly above or below its domain $\boldsymbol{D}$ in the xy-plane (Figure 1).

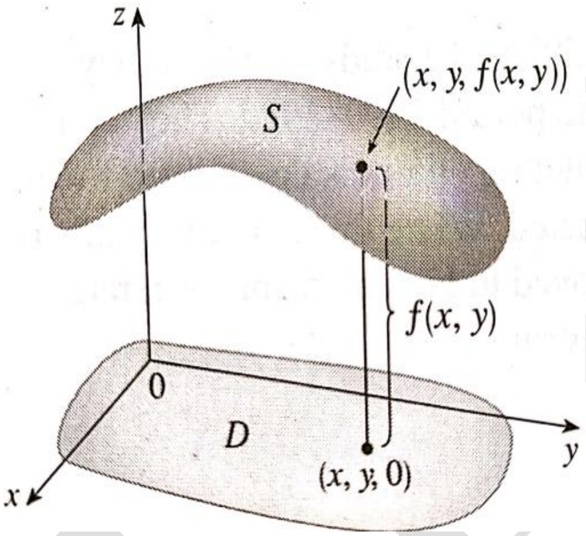


Figure 1

Definition: A function $\boldsymbol{f}$ of two variables is a rule that assigns to each ordered pair of real numbers $(\boldsymbol{x}, \boldsymbol{y})$ in a set $\boldsymbol{D}$ a unique real number denoted by $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$. The set $\boldsymbol{D}$ is the domain of $\boldsymbol{f}$ and its range is the set of values that $\boldsymbol{f}$ takes on, that is, $\{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}\}$.

Example: The function $g(x, y) = \sqrt{9 - x^2 - y^2}$ is represented graphically to enhance the understanding of the function. The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain, $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 2).

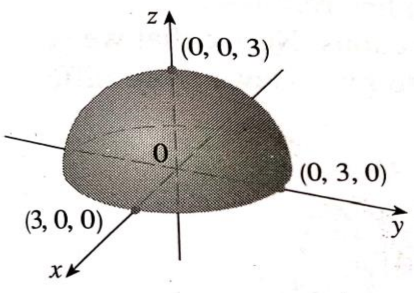


Figure 2

Functions of three variables:

Functions of three or more variables are defined analogously to functions of two variables.

For example, a function of three variables is a rule of correspondence that assigns to each ordered triple of real numbers (x, y, z) of a subset of 3-space one and only one number w in the set $\mathbb{R}$ of real numbers.

We write,
$w = F(x, y, z)$. Although we cannot draw a graph of a function of three variables $w = F(x, y, z)$,

"We can draw the surfaces defined by" $F(x, y, z) = c$, for suitable values of c.
These surfaces are called **level surfaces.** (level surfaces are usually not level).

The change in temperature T which depends on two parameters like pressure and entropy can be analyzed in two ways as, the change of temperature with respect to pressure at constant entropy is related to the coefficient of thermal expansion. Similarly, the change of temperature T with respect to entropy S at constant pressure is related to the concept of specific heat.

We observe the following concept from this case study.

Partial Derivatives:

The derivative of a function of one variable $\boldsymbol{y = f(x)}$ is given by

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

In exactly the same manner, we can define a derivative of a function of two variables with respect to each variable.

If $z = f(x, y)$, is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Tree representation of Ordinary Derivative (OD) and Partial Derivative (PD):

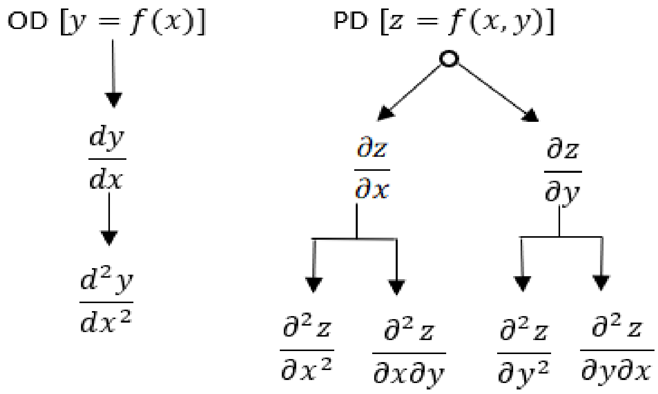


Figure 3

Rule for finding partial Derivatives of $z = f(x, y)$

- To find f_x, regard y as constant and differentiate $f(x, y)$ with respect to x.

Example: If $f(x, y) = x^5 - 2x^2y^3$

Solution: $\frac{\partial f}{\partial x} = 5x^4 - 4xy^3$

- To find f_y, regard x as constant and differentiate $f(x, y)$ with respect to y.

Example: If $f(x, y) = x^5 - 2x^2y^3$

Solution: $\frac{\partial f}{\partial y} = 6x^2y^2$

Notations for Partial Derivatives:

If $Z = f(x, y)$, we write first order Partial derivatives as;

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}; \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$$

and second order Partial derivatives are:

$$f_{xx}(x, y) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right); \qquad f_{yy}(x, y) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

The mixed second-order partial derivatives are:

$$f_{xy}(x, y) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right); \quad f_{yx}(x, y) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Note: $f_{xy} = f_{yx}$

Geometrical Interpretation of Partial Derivatives:

At a specific point $(a,\ b)$ on the surface defined by $f\ (x,\ y)$ is geometrically represented as shown in Figure 4.

The partial derivative $\frac{\partial f}{\partial x}$ gives the slope of the tangent line to the surface at that point in the x-direction. This tangent line lies in the plane formed by fixing y constant.

Similarly, the partial derivative $\frac{\partial f}{\partial y}$ gives the slope of the tangent line to the surface at that point in the y-direction. This tangent line lies in the plane formed by fixing x constant.

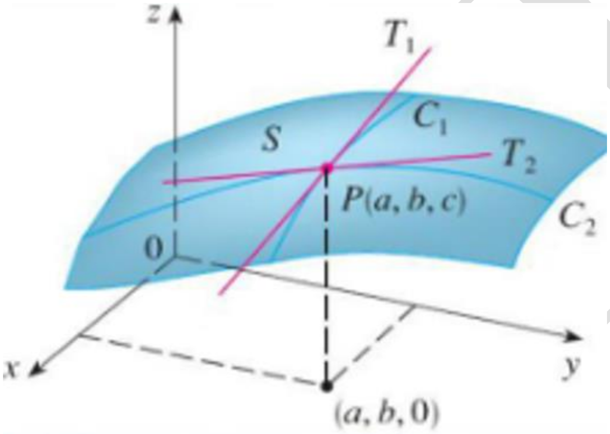


Figure 4

Chain Rule in Partial Derivatives:

The Chain Rule for functions of one variable state that if $y = f\ (u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then the derivative of the composite function is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

For a composite function of two variables $z = f\ (x,\ y)$, the Chain Rule for functions of two variables is summarized as follows:

If $z = f\ (x, y)$ is differentiable and $x = g(t)$ and $y = h(t)$ have continuous first partial derivatives, then

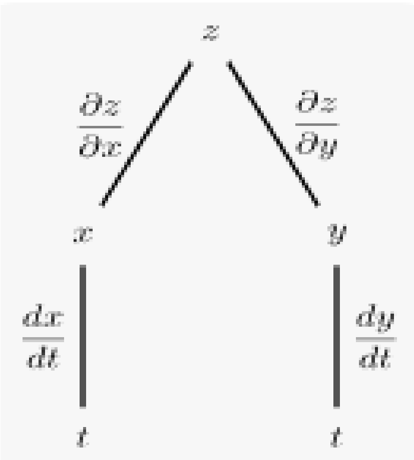
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$$

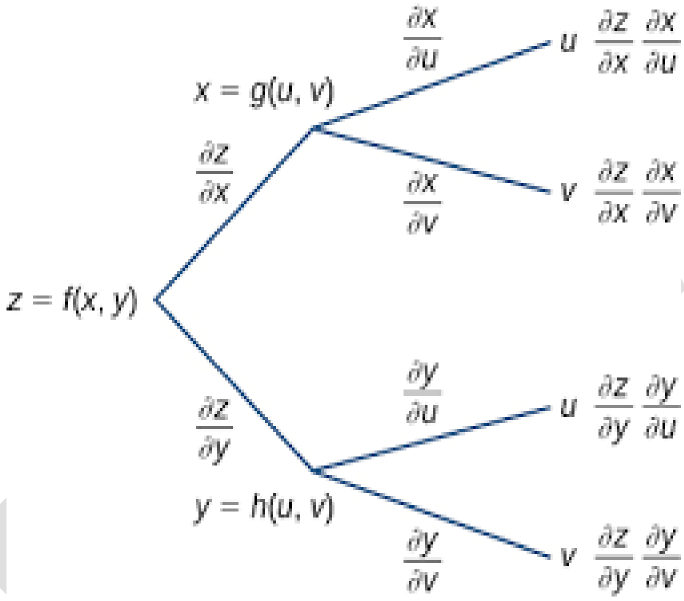
If $z = f\ (x, y)$ is differentiable and $x = g(u,\ v)$ and $y = h(u,\ v)$ have continuous first partial derivatives, then

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial v}$$

Tree Diagram for chain rule for function of two variables:





Applications of Partial Differentiation:

Environmental Science:

- Ecosystem Dynamics: Analyzing the impact of environmental factors on species populations.

Physics and Engineering:

- Used to model and analyse physical systems with multiple variables, such as heat conduction, fluid flow, and wave propagation.

Economics and Finance:

- Utilized in optimization problems, like maximizing profit or minimizing cost functions in economics.

Example 1: If $f(x,y)=x^3+x^2y^3-2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution: Holding y constant and differentiating with respect to x, we get

$$f_x(x,y)=3x^2+2xy^3$$

$$f_x(2,1)=3(2)^2+2(2)(1)^3=16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$$

Example 2: If $f(x,\ y)\ = sin\left(\frac{x}{1+y}\right)$ calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: Using the Chain Rule, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right).\ \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right).\frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right).\ \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right).\frac{x}{(1+y)^2}$$

Example 3: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3\ +\ y^3\ +\ z^3\ +\ 6xyz\ =\ 1$

Solution: To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$,we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Rule for finding Partial Derivatives of $u = f(x, y,\ z)$:

- To find u_x, regard $y,\ z$ as constant and differentiate $f(x,y,z)$ with respect to x.
- To find u_y, regard $x,\ z$ as constant and differentiate $f(x,y,z)$ with respect to y.
- To find u_z, regard x,y as constant and differentiate $f(x,y,z)$ with respect to z.

Example: Find first, second, third and mixed partial derivative of $u = x^3 + y^3 + z^3 + 3xyz$ with respect to $x,\ y\ and\ z$.

Solution: $u_x = 3x^2 + 3yz$; $u_y = 3y^2 + 3xz$; $u_z = 3z^2 + 3xy$

$$u_{xx} = 6x;\ u_{yy} = 6y;\ u_{zz} = 6z;\ u_{xy} = 3z;\ u_{yz} = 3x;\ u_{zx} = 3y$$

$$u_{xxx} = 6;\ u_{yyy} = 6;\ u_{zzz} = 6;$$

$$u_{xxy} = u_{xxz} = u_{yyx} = u_{yyz} = u_{zzx} = u_{zzy} = 0$$

$$u_{xyz} = u_{yzx} = u_{zxy} = 3$$

Example: Consider a metal rod where temperature $T(x,t) = 3x^2t + 5xt^3$ depends on both position x and time t. Find the rate of change of temperature along with the rod with respect to position and time at (2,1).

Solution: Change of temperature along with the rod with respect to position

$\frac{\partial T}{\partial x} = 6xt + 5t^3$; At (2,1), $\frac{\partial T}{\partial x} = 12 + 5 = 17$.

Change of temperature along with the rod with respect to time

$\frac{\partial T}{\partial t} = 3x^2 + 15xt^2$; At (2,1), $\frac{\partial T}{\partial t} = 12 + 30 = 42$.

Practice Problems:

1. Find the first, second and mixed partial derivatives of the function.

(i) $f(x, y) = y^5 - 3xy$

(ii) $f(x, y) = x^4y^3 + 8x^2y$

2. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ of $w = \frac{e^v}{u+v^2}$

3. A production function $P(L, K) = 2L^3 + LK + K^2$ may represent the output P as a function of labour L and capital K. Find the change in output concerning changes in each input.

Ans: $\frac{\partial P}{\partial L} = 6L^2 + K$: $\frac{\partial P}{\partial K} = L + 2K$

Home Work Problems:

1. The electric field $E(x, y, z) = x^2y^3z^4$ may be a function of spatial coordinates. Find the rate of change of the electric field in each direction with respect to the spatial coordinates.
2. Find the first, second and mixed partial derivatives of the function.
   $(i) f(x, t) = e^{-t}(\cos \pi x)$ $(ii) f(x, t) = \sqrt{x}(\ln t)$ $(iii)\ z = (2x + 3y)^{10}$

Imagine you are running small business where your profit (f) depends on both the number of products sold (x) and advertise expenses (y). The profit function is influenced by both the variables x (number of products sold) and y (advertise expenses). We need to know the total change in profit. This understanding can be valuable for making informed business decisions in response to changes in different variables.

From this case study we observe the concept of total derivatives:

Total Derivatives:

Definition: The total derivative of $f(x, y)$ with respect to x and y is given by,

$df = \frac{\partial f}{\partial x} \times dx + \frac{\partial f}{\partial y} \times dy.$

The total derivative of $f(x, y)$ with respect to t, where x, y are functions of t then:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

Example 1: Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function $f(x, y) = x^2 + 2xy - y^2$. Evaluate the total differential df at the point $(2, -1)$.

Solution: Let $f(x, y) = x^2 + 2xy - y^2$

We know that $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ were

$\frac{\partial f}{\partial x} = 2x + 2y$ and $\frac{\partial f}{\partial y} = 2x - 2y$

Therefore, $df = (2x + 2y)dx + (2x - 2y)dy$

Then df at $(2, -1)$ is $df = (2 \times 2 + 2 \times (-1))dx + (2 \times 2 - 2(-1))dy$

$df = (4 - 2)dx + (4 + 2)dy$

$df = 2dx + 6dy$

Example: In fluid dynamics, the velocity potential function $f(x, y) = x^2 + 2y^3 x$ represents a scalar field that describes the velocity potential of a fluid flow and $x = 3t^2$, $y = 2t$ represent any two perpendicular spatial directions, such as horizontal and vertical coordinates. Find the rate at which the velocity potential changes concerning time, indicating how the fluid flow evolves temporally at a specific spatial location.

Solution: The rate at which the velocity potential changes concerning time is,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

$$= (2x + 2y^3)(6t) + 6y^2x(2)$$

$$= [2(3t^2) + 2(2t)^3](6t) + 6(2t)^2(3t^2)(2)$$

$$= 36t^3 + 240t^4$$

Practice Problems:

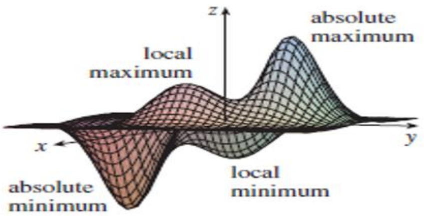
1. Determine $\frac{df}{dt}$ for the velocity potential function $f(x, y) = ln(x) + 4y^2$ where $x = e^{2t}$ *and* $y = \sqrt{t}$.
2. For the velocity potential function $f(x, y) = e^{2x} + siny$, calculate $\frac{df}{dt}$ at $x = t$, $y = 2t$.
3. An electric potential function $V(x, y) = 3y^2 + x^3y + 4x^3$ in a two-dimensional space, the variables $x = 2t$ and $y = e^t$ typically represent spatial coordinates. Find the rate of change of electric potential energy of a particle changes concerning time at $t = 1$.

   Ans: $\frac{dV}{dt} = 24t^2e^t + 96t^2 + 6e^{2t} + 8t^3e^t$. At $t = 1$, $\frac{dV}{dt}$ =227.32.

Home Work Problems:

1. Given the velocity potential function $f(x, y) = 2cosx + +2y^3sin2x$, and find $\frac{df}{dt}$ at $x = 3t$ *and* $y = 2t$.
2. If $f(x, y) = x^2 + 2y^3x$ represents a production function, where labour (x) and capital (y) are inputs. Find the rate of change of production relating to time (t) in the production process as labour $x = 2t + 1$ and capital $y = t + 3$ changes.

Look at the hills and valleys in the graph of f shown in the following figure. There are two points (a, b) where f has a local maximum, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the absolute maximum. Likewise, f has two local minima, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the absolute minimum.



Maximum and Minimum Values

Definition: A function of two variables has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a local maximum value.

If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and $f(a, b)$ is a local minimum value. If the inequalities in definition hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$ then $f_x(x,y) = 2x - 2$ and $f_y(x,y) = 2y - 6$ $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = 2y - 6 = 0$ gives $x = 1$ and $y = 3$. Therefore, $f(1,3) = 4$ is a local minimum and in fact it is the absolute minimum of f. Geometrically, graph f is the elliptic paraboloid with vertex (1, 3, 4) as shown in Figure 5.

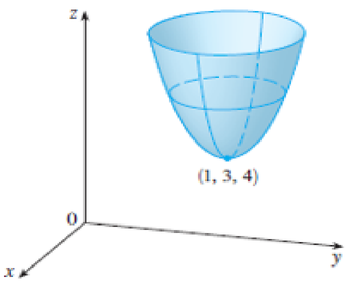


Figure 5. z= $x^2 + y^2 - 2x - 6y + 14$

Let $f(x, y) = y^2 - x^2$; $f_x(x,y) = -2x = 0$ and $f_y(x,y) = 2y = 0$ gives $x = 0$ and $y = 0$. Therefore, $f(0,0) = 0$ can't be an extreme value for f, so f has no extreme value. Above example illustrates the fact that a function need not have a maximum or minimum value at a critical point (0, 0). The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, as shown in Figure 6 which has horizontal tangent plane (z=0) at the origin. Observe that $f(0,0) = 0$ is a *maximum* in the direction of the x-axis but a *minimum* in the direction of the y-axis. Near the origin the graph has the shape of a saddle and so (0, 0) is called a ***Saddle point* of f.**

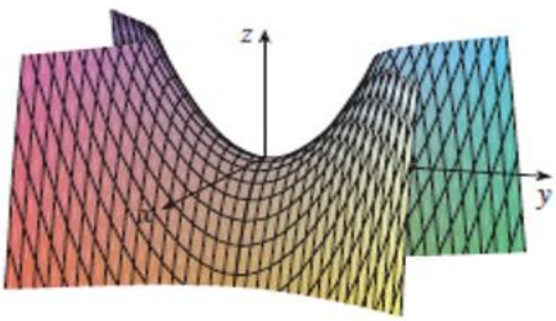


Figure 6. $z = y^2 - x^2$

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test, is analogous to the *Second Derivative Test* for functions of one variable.

Second Derivatives Test: Suppose the second partial derivatives of f are continuous on a disk with center $(a,\ b)$, and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [that is, (a, b) is a critical point of f].

Let $AC - B^2 = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

Where $A = f_{xx}(a,b)$; $C = f_{yy}(a,b)$ and $B = f_{xy}(a,b)$

(a) If $AC - B^2 > 0$ and $A > 0$, then $f(a,\ b)$ is a local minimum.

(b) If $AC - B^2 > 0$ and $A < 0$, then $f(a,\ b)$ is a local maximum.

(c) If $AC - B^2 < 0$, then $f(a,\ b)$ is not a local maximum or minimum.

Note 1: In case (c) the point $(a,\ b)$ is called a **saddle point** of and the graph of f crosses its tangent plane at $(a,\ b)$.

Note 2: If $AC - B^2 = 0$, the test gives no information: f could have a local maximum or local minimum at $(a,\ b)$, or $(a,\ b)$ could be a saddle point of f.

Example 1: Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution: We first locate the critical points and equating it to zero we obtain the equation:

$f_x = 4x^3 - 4y = 0$ and $f_y = 4y^3 - 4x = 0$

To solve these equations, we substitute $y = x^3$ from the first equation into the second one. This gives

$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$ so, there are three real roots: 0, 1, -1. The three critical points are (0, 0) (1, 1) and (-1, -1).

Next, we calculate second partial derivative and $D(x,\ y)$

	(0, 0)	(1, 1)	(-1, -1)
A=$f_{xx} = 12x^2$	0	$12 > 0$	$12 > 0$
C=$f_{yy} = 12y^2$	0	12	12

B=$f_{xy} = -4$	-4	-4	-4
$D = f_{xx}f_{yy} - (f_{xy})^2$	$-16 < 0$	$128 > 0$	$128 > 0$
Result	No information	Local minimum	Local minimum

The graph of f is shown in Figure 7.

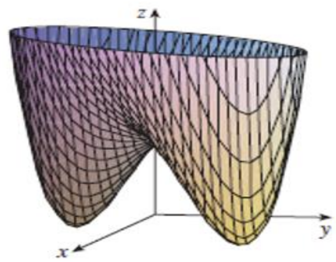


Figure 7. $z = x^4 + y^4 - 4xy + 1.$

Example: A Storage unit without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

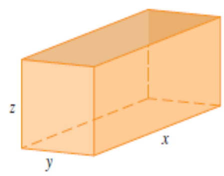


Figure 8.

Solution: Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 8. Then the volume of the box is

$$V = xyz.$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

We get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

$$\frac{\partial V}{\partial x} = \frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0 \text{ and } \frac{\partial V}{\partial x} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0$$

$$12 - 2xy - x^2 = 0 \text{ and } 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$.

If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2,\ y = 2$, and $z = \frac{12-2(2)}{[2(2+2)]} = 1$.

Then $V = (2)(2)(1) = 4$, has maximum volume $4m^3$.

Practice problems:

1. Find the extreme value of the function $f(x,\ y) = xy(a - x - y)$.

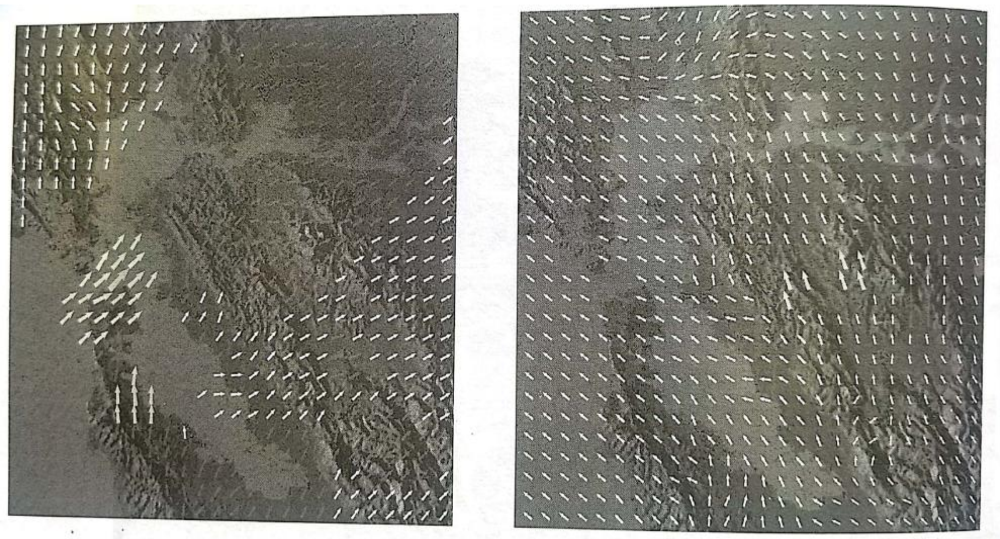
   Ans: Max. Value of $f = \frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$

2. The temperature T at a point $(x,\ y,\ z)$ in space $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

   Ans: Highest temperature=50 at $\left(\frac{1}{2}, \frac{1}{2}\right)$

Home Work problems:

1. Find the maximum value of the function $f(x,\ y) = x^3y^2(1 - x - y)$.
2. A rectangular box, open at the top, is to have a volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.
3. Consider a dome shaped building where the height in meters above ground level at any point $(x,\ y)$ on the dome's surface is given by the function $f(x,\ y) = 50 - (x^2 + y^2)$. Determine the maximum height of the dome. Represent the dome geometrically and compare the mathematical solution obtained with the graph justifying the solution.



(a) 6:00 PM, March 1, 2010 (b) 6:00 AM, March 1, 2010

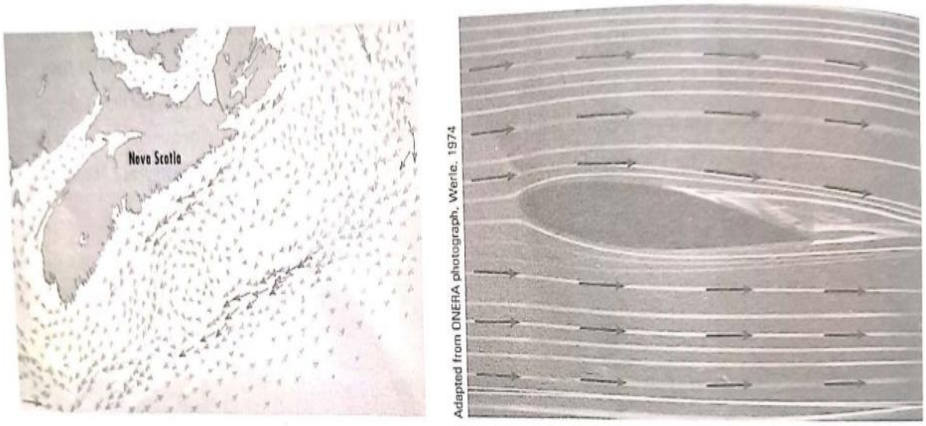
Figure 9: Velocity vector fields showing San Francisco Bay wind patterns

Vector Fields:

The vectors in Figure 9 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge.

Part (b) shows the very different wind pattern 12 hours earlier. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a velocity vector field.

Other examples of velocity vector fields are illustrated in Figure 10, Ocean currents and flow past an air foil.



(a) Ocean currents off the coast of Nova Scotia (b) Airflow past an inclined airfoil

Figure 10: Velocity vector fields showing San Francisco Bay wind patterns

Vector Calculus:

Here we study the calculus of vector fields. These are functions that assign vectors to points in the space.

Vector: A quantity that has both magnitude and direction is called a vector.

Ex: Velocity, Acceleration, Force etc…

Scalar: A quantity which has magnitude, but no direction is called a scalar.

Ex: Mass, Volume, Density.

Position Vector: If a point 'O' is fixed as origin in space (or plane) and 'P' is any point then $\overrightarrow{OP}$ is called the position vector of P w.r.t 'O'.

Definition: Let D be a set in $\mathbb{R}^2$ (a plane region) or $\mathbb{R}^3$. A vector field on $\mathbb{R}^2$(or $\mathbb{R}^3$) is a function F that assigns to each point (x, y) [or (x, y, z)] in D a two-dimensional vector $F(x, y)$ [or a three-dimensional vector $F(x, y, z)$].

The best way to picture a vector field is to draw the arrow representing the vector $F(x, y)$ starting at the point (x, y). In Figure 11 and Figure 12 for a few representative points in D a vector $F(x, y)$ and[$F(x, y, z)$] are drawn.

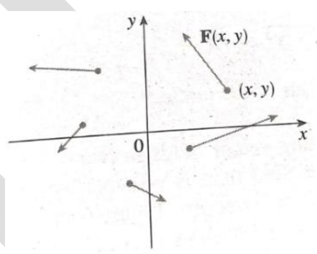


Figure 11: Vector field on $\mathbb{R}^2$

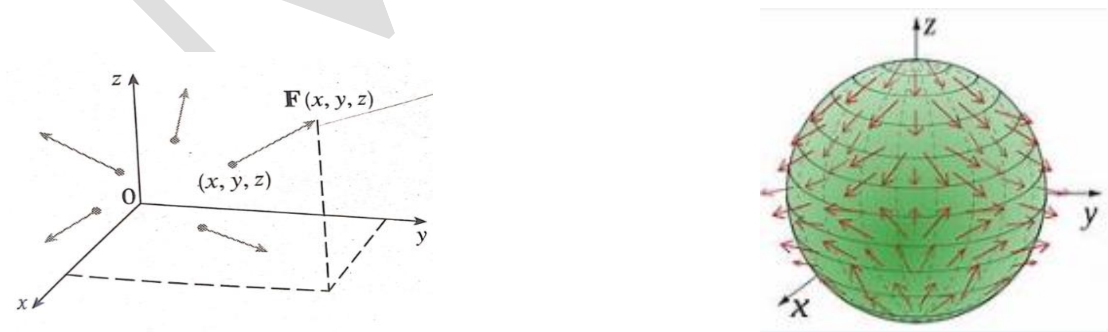


Figure 12: Vector field on $\mathbb{R}^3$

We can express it in terms of its component functions P, Q, and R as

$$F(x,\ y) = P(x,y)i + Q(x,\ y)j = P\boldsymbol{i} + Q\boldsymbol{j}$$

$$\text{and}\ \ F(x,y,z) = P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Example: A vector field on $\mathbb{R}^2$ is defined by $F(x,y) = -y\boldsymbol{i} + x\boldsymbol{j}$. Describe F by sketching some of the vectors $F(x,\ y\ )$ as in Figure 13.

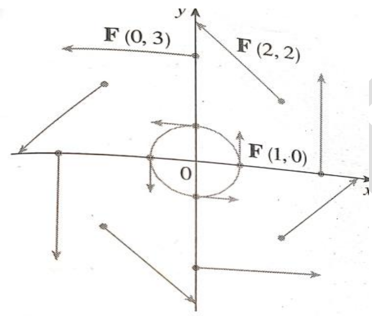


Figure 13: $F(x,y) = -yi + xj$

Solution: Since $F(1{,}0) = \boldsymbol{j}$, we draw the vector $\boldsymbol{j} = (0,\ 1)$ starting at the point (1, 0) in Figure 13. Since $F(0,\ 1\ ) = -\boldsymbol{i}$, we draw the vector <-1,0 > with starting point (0, 1). Continuing in this way, we calculate several other representative values of $F(x,\ y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 13.

(x, y)	F (x, y)	(x, y)	F (x, y)
(1,0)	<0,1>	(-1, 0)	<0, -1>
(2, 2)	<-2,2>	(-2, -2)	<2, -2>
(3,0)	<0,3>	(-3, 0)	<0, -3>
(0,1)	<-1,0>	(0, -1)	<1, 0>
(-2,2)	<-2, -2>	(2, -2)	<2, 2>
(0,3)	<-3,0>	(0, -3)	<3, 0>

Observe the sketch is shown in Figure 14. of $F(x,\ y,\ z) = z\boldsymbol{k}$. Notice that all vectors are vertical and point upward above the $xy-plane$ or downward below it. The magnitude increases with the distance from the $xy-plane$.

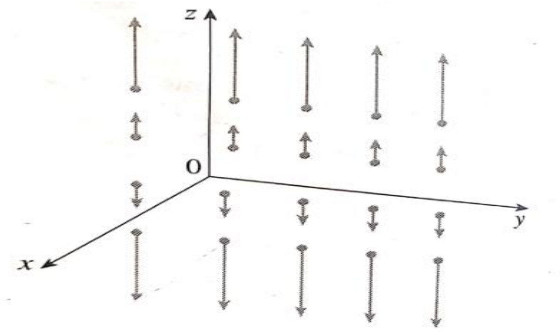


Figure 14: $F(x,y,z) = z\boldsymbol{k}$

Gradient:

A vector differential operator is defined as;

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} \quad (or) \quad \nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

and is applied to a differentiable function $z = f(x, y)$ or $w = F(x, y, z)$, we say that the vectors

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

are the **Gradients** of the respective functions. The symbol ∇, an inverted capital Greek delta, is called **"del" or "nabla"** is a differential operator. The vector ∇f is usually read "$\boldsymbol{grad\ f}$."

Example 1: Find the gradient of the level surface of $F(x, y, z) = x^2 + y^2 + z^2$ passing through (1, 1, 1).

Solution: Since $F(1, 1, 1) = 3$, the sphere $x^2+y^2+z^2=3$.

The Gradient of the function is

$\nabla F(x, y, z) = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$,

$\nabla F(1, 1, 1) = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

The level surface and ∇F (1, 1, 1) are illustrated in Figure 16.

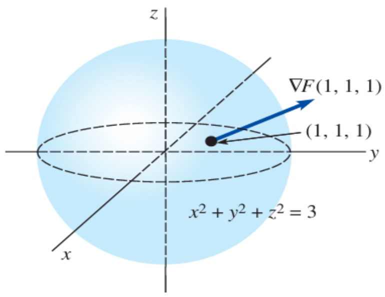


Figure 16.

Example 2: Given $f = x^3 + y^3 + z^3 - 3xyz$ find ∇f(gradient of f) at the point $(1, -2, 3)$.

Solution: $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$

$$= \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial x}\boldsymbol{i} + \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial y}\boldsymbol{j} + \frac{\partial(x^3+y^3+z^3-3xyz)}{\partial z}\boldsymbol{k}$$

$$\nabla f = (3x^2 - 3yz)\boldsymbol{i} + (3y^2 - 3xz)\boldsymbol{j} + (3z^2 - 3xy)\boldsymbol{k}$$

$$\nabla f(1, -2, 3) = 21\,\boldsymbol{i} + 3\,\boldsymbol{j} + 33\,\boldsymbol{k}$$

Suppose you are an aerospace engineer working on optimizing the design of an aircraft wing to improve fuel efficiency. One crucial aspect is understanding the airflow around the wing.

The Divergence of the airflow, in this case, represents the spreading or spreading out of air as it flows around the wing. Divergence can lead to changes in air pressure and affect the lift and drag forces on the aircraft.

Divergence:

The Divergence of a vector field $\vec{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, is a scalar function.

Then, $\boldsymbol{div\vec{F}} = \nabla \cdot \vec{\boldsymbol{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Observe that $div\, F$ can also be written in terms of the del operator as

$$div\vec{F} = \nabla.\vec{F} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right).\,[P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}]$$

$$= \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z).$$

Example: If $\vec{\boldsymbol{A}} = xz^3\,\boldsymbol{i} - 2x^2y^2\,\boldsymbol{j} + 2yz^4\,\boldsymbol{k}$ find $\boldsymbol{\nabla} \cdot \vec{\boldsymbol{A}}$ at $(-1, 2, 1)$.

Solution: $\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right).\,(xz^3\,\boldsymbol{i} - 2x^2y^2\,\boldsymbol{j} + 2yz^4\,\boldsymbol{k})$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2y^2) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 4x^2y + 8yz^3$$

$$\therefore \nabla.\vec{A}\ at\ (-1, 2, 1)\ is\ 9.$$

If we are given a function $V(x, y, z)$ that defines a vector field, along with some specific point in space (x_0, y_0, z_0), how much does a fluid flowing along the vector field rotate at the point (x_0, y_0, z_0)?

The video (https://youtu.be/kXffMzSax7U) shows a simulation of what this might look like. A sample of fluid particles, shown as blue dots, will flow along the vector field. This means that at any given moment, each dot moves along the arrow it is closest to. Focus in particular on what happens in the four circled regions.

Curl:

The Curl of a vector field $\vec{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$, is the vector field.

$$Curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}.$$

Example: If $\vec{A} = xz^3\ \boldsymbol{i} - 2x^2 y^2\ \boldsymbol{j} + 2yz^4\ \boldsymbol{k}$ find $\nabla \times \vec{A}$ and $\nabla \cdot (\nabla \times \vec{A})$.

Solution: $$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2 y^2 & 2yz^4 \end{vmatrix} = 2z^4 i + 3xz^2 j - 4xy^2\ \boldsymbol{k}$$

$\nabla \times \vec{A}$ at (-1, 2, 1) is $2i+3j-16\boldsymbol{k}$

$$\nabla \cdot (\nabla \times \vec{A}) = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right).(2z^4 i + 3xz^2 j - 4xy^2 k)$$

$$= \frac{\partial}{\partial x}(2z^4) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-4xy^2) = 0.$$

Practice Problems:

1. Find $div\vec{f}$ and $Curl\vec{f}$ of $\vec{f} = 2x^2z\boldsymbol{i} - 10xyz\boldsymbol{j} + 3xz^2\boldsymbol{k}$ at (1, -1, 1).

Ans: $\nabla \cdot A$ at $(1, -1, 1) = 0$, $\nabla \times A \rightarrow$ at $(1, -1, 1) = -10i - j + 10k$

2. If $\vec{F} = \nabla(x\, y^3\, z^2)$ find **div** $\vec{F}$ and Curl$\vec{F}$ at the point $(1, -1, 1)$.

Ans: div F at $(1, -1, 1) = -8$, Curl $F = 0$.

3. If $\phi = x^2z + e^{\frac{y}{x}}$, Find $\nabla\phi$ Ans: $[2xz + e^{\frac{y}{x}}\left(\frac{-y}{x^2}\right)]\boldsymbol{i} + e^{\frac{y}{x}}\left(\frac{1}{x}\right)\boldsymbol{j} + x^2\boldsymbol{k}$

4. If $\vec{f} = x^2y\ \boldsymbol{i} - (z^3 - 3x)\ \boldsymbol{j} + 4y^2\ \boldsymbol{k}$, Find $div\vec{f}$ and $Curl\vec{f}$

Ans: $div\vec{f} = 2xy$ and $Curl\vec{f} = (8y + 3z^2)\boldsymbol{i} + (3 - x^2)\boldsymbol{k}$

5. If $\vec{f} = (3x + 2z^2)\,\boldsymbol{i} + \left(\frac{x^3y^2}{z}\right)\boldsymbol{j} - (z - 7x)\,\boldsymbol{k}$, Find $div\vec{f}$ and $Curl\vec{f}$.

Ans: $div\vec{f} = 2 + \left(\frac{2x^3y}{z}\right)$ and $Curl\vec{f} = \frac{-x^3y^2}{z^2}\boldsymbol{i} - (\mathbf{7} - 4z)\boldsymbol{j} + \left(\frac{3x^2y^2}{z}\right)\boldsymbol{k}$

6. If $\vec{f} = (2y - \cos x)\,\boldsymbol{i} - (z^2e^{3x})\,\boldsymbol{j} + (x^2 - 7z)\,\boldsymbol{k}$, Find $div\vec{f}$ and $Curl\vec{f}$

Ans: $div\vec{f} = sinx - 7$ and $Curl\vec{f} = \mathbf{2z}e^{3x}\boldsymbol{i} - \mathbf{2x}\boldsymbol{j} - (\mathbf{3}z^2e^{3x} + \mathbf{2})\boldsymbol{k}$

Interpretation of Divergence:

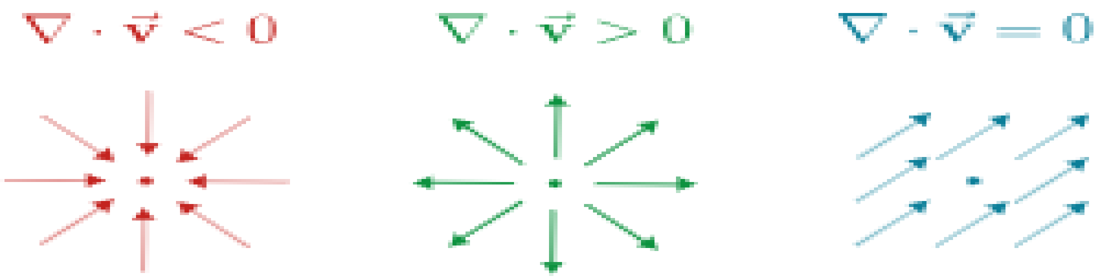


Figure (a) Figure (b) Figure (c)

(i) Figure (a), $\nabla \cdot \vec{v} < 0$ vector field becomes **more dense** at the point. A point at which the flux is directed inward has negative divergence, and is often called a **"sink"** of the field.

(ii) Figure (b), $\nabla \cdot \vec{v} > 0$ vector field becomes **less dense** at the point. Also, a point at which the vector is outgoing has positive divergence, and is often called a **"source"** of the field.

(iii) Figure (c), $\nabla \cdot \vec{v} = 0$ vector field **density stays constant.** Also, when $\nabla \cdot \vec{v} = 0$, $\vec{v}$ is said to be **Solenoidal vector**.

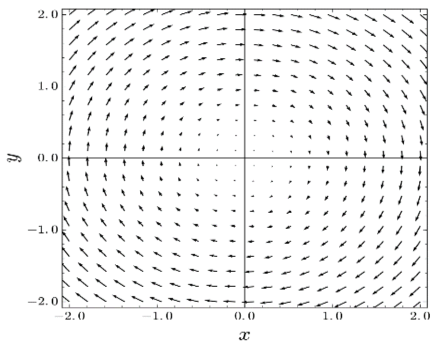
It can also be interpreted in the case of flow of fluid as:

i) when $\nabla \cdot \vec{v} < 0$ then **rate of inflow > rate of outflow.**

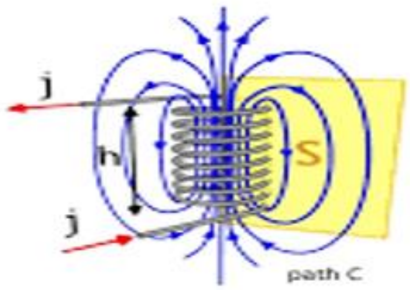
ii) when $\nabla \cdot \vec{v} > 0$ then **rate of inflow < rate of outflow.**

iii) when $\nabla \cdot \vec{v} = 0$, then **rate of inflow = rate of outflow.**

Graphical representation of Solenoidal:



Solenoidal of vector field



Solenoids in Magnetostatics

Example 1: Is $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is solenoidal vector?

Solution: $\nabla.\vec{V}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right).\left(\frac{x\boldsymbol{i}}{x^2+y^2}+\frac{y\boldsymbol{j}}{x^2+y^2}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)+\frac{\partial}{\partial z}(0)$$

$$=\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}=0$$

Hence, $\vec{F}$ is solenoidal vector.

Example 2: Consider the velocity field $V(x,\ y,\ z)=xz\boldsymbol{i}-xy\boldsymbol{j}+yz\boldsymbol{k}$. Interpret the divergence of a vector field at $(2,\ 1,\ 1),\ (2,\ -1,\ 1)\ and\ (-2,\ 1,\ 1)$.

Solution: $\nabla.\vec{V}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right).\ (xz\boldsymbol{i}-xy\,\boldsymbol{j}+yz\boldsymbol{k})$

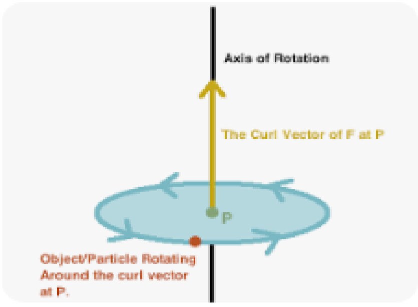
$$=\frac{\partial}{\partial x}(xz)+\frac{\partial}{\partial y}(-xy)+\frac{\partial}{\partial z}(yz)$$

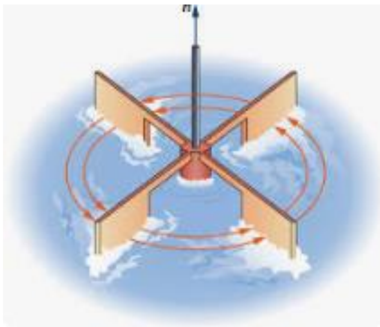
$$=z-x+y$$

Hence, $\vec{V}$ is solenoidal at $(2,\ 1,\ 1)$, $\vec{V}$ is sink at $(2,\ -1,\ 1)$ and $\vec{V}$ is source at $(-2,\ 1,\ 1)$.

Geometrical interpretation of Curl:

Curl is an operator which measures **rotation** in a fluid flow indicated by a three-dimensional vector field.





If there is no rotation then it is said to be **irrotational** i.e. Curl of a vector is zero.

Example1: The fluid behaviour of a fluid motion $\vec{F} = (x+y)i + (y+z)j - (x+z)k$ is known to engineers. Find the local rotation or spin of fluid elements within a fluid flow to provide the information about the rotation of fluid particles at a particular point (2, -1, 3) in the flow.

Solution: $\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (y+z) & -(x+z) \end{vmatrix} = -i+j-k$

Example2: Find a, b, c so that $\vec{\boldsymbol{F}} = (x+2y+az)\boldsymbol{i} + (bx-3y-z)\boldsymbol{j}$ +(4x+cy+2z) $\boldsymbol{k}$ is Irrotational.

Solution: $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$

$$= (c+1)i + (a-4)\boldsymbol{j} + (b-2)\boldsymbol{k}$$

Evidently, $\nabla \times \vec{F} = 0$ whenever $a = 4,\ b = 2$ and $c = -1$

Practice Problems:

1. Find div$\vec{F}$ and Curl$\vec{F}$ Where $\nabla\vec{F}$ ($x^3 + y^3 + z^3 - 3xyz$).

Ans: $div\vec{F} = 6(x+y+z)$, Curl $\vec{F} = 0$.

2. Examine that in the conservation of energy of fluid motion $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is the fluid particles rotates or not? Ans: Irrotational.

3. Find the value of the constant ' $\boldsymbol{a}$ ' such that the vector field, $\vec{\boldsymbol{F}} = (\boldsymbol{a}xy - z^3)\boldsymbol{i} + (\boldsymbol{a}-2)\,x^2\boldsymbol{j} + (1-\boldsymbol{a})\,xz^2\boldsymbol{k}$ is Irrotational.

Ans: $\therefore a = 4$

4. Show that $\vec{f} = (z + siny)i + (xcosy - z)\boldsymbol{j} + (x - y)k$ is Irrotational.

5. Consider the stress vector field $T(x,\ y,\ z) = -yx\boldsymbol{i} + xz\boldsymbol{j} + z^2\boldsymbol{k}$ within a solid material. Is the stress distribution across the material is going to be spread or sinking at $(-2,\ 1,\ -1)$?

Ans: $-3.$ Stress distribution is going to sink at the given point.

Home Work Problems:

1. Show that $\vec{\boldsymbol{F}}$= $(\boldsymbol{z} + \sin \boldsymbol{y})\boldsymbol{i} + (\boldsymbol{x}\cos \boldsymbol{y} - \boldsymbol{z})\boldsymbol{j} + (\boldsymbol{x} - \boldsymbol{y})\boldsymbol{k}$ is Irrotational.

2. Find the value of the constant '$\boldsymbol{a}$' and 'b' such that the vector field, $\vec{\boldsymbol{F}}$= $(axy + z^3)\boldsymbol{i}$ + $(3x^2 - z)\ \boldsymbol{j}$ + $(bxz^2 - y)\boldsymbol{k}$ is Irrotational.

Ans: $a = 6,\ b = 3.$

3. In structural mechanics, predicting and analysing the structural integrity of buildings, depends on rotation of structures having displacement vector field $\vec{F} = x^2y\boldsymbol{i} - y^2z\boldsymbol{j} + z^2x\boldsymbol{k}$. Find the curl of vector field $\vec{F}$ to analyse the structural integrity of building.

4. To design and analyse the electromagnetic devices such as transformers having the electric field $\vec{F} = \frac{x}{y}\boldsymbol{i} - \frac{y}{z}\boldsymbol{j} + \frac{z}{x}\boldsymbol{k}$, find the curl of $\vec{F}$ at $(1, -2,\ 3)$.

Geometrical Interpretation of the Gradient

∇F is normal (perpendicular) vector to the level surface at P which is shown in the following Figure 17.

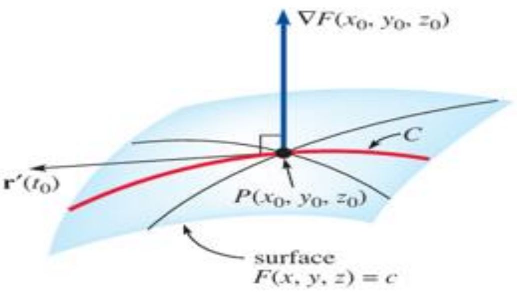


Figure 17: Gradient is normal to the level surface at *P*

Physical interpretation of Gradient in various Engineering fields:

Physics: In physics, especially in the context of fields like heat, fluid dynamics, or electromagnetism, the gradient represents the spatial rate of change of a physical quantity. For example, in a temperature field, the gradient of temperature indicates how quickly temperature changes in space.

Graphics and Imaging: In image processing, the term gradient is used to describe the change in intensity or color in an image. The gradient provides information about the edges or boundaries within an image.

Machine Learning: In Machine Learning, particularly in the context of optimization algorithms, the gradient is often used to denote the vector of partial derivatives of a loss function with respect to the model parameters. It guides the update of model parameters during the training process, helping to find the minimum (or maximum) of the loss function.

The weather map in Figure shows a contour map of the temperature function for the states of California and Nevada at 3:00 PM on a day in October.

The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno;

T_y is the rate of change of temperature if we travel north.

But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction?

In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivative: The Directional Derivative of $z = f(x, y)$ in the direction of a unit vector $u = \cos\theta\, \boldsymbol{i} + \sin\theta \boldsymbol{j}$ is

$$D_u f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$

provided the limit exists.

Observe that above equation is truly a generalization of partial differentiation, since

$\theta = 0$ implies that $D_x f(x, y) = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial z}{\partial x}$ and

$\theta = \frac{\pi}{2}$ implies that $D_y f(x, y) = \lim_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial z}{\partial y}$

Computing a Directional Derivative:

If $z = f(x, y)$ is a differentiable function of x and y and $u = \cos\theta\, i + \sin\theta\, j$, then

$$D_u f(x, y) = \nabla f(x, y).u$$

(Or) Directional Derivative of f along $\hat{a}$ is $\nabla f \cdot \hat{a}$

Maximizing the Directional Derivatives:

Suppose we have a function of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answer is Maximizing the Directional Derivative denoted as $|\nabla f|$.

Unit Normal Vector:

If $f\ (x,\ y,\ z) = c$ represents a surface, then the unit normal vector to the surface is defined by

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

Example 1: Find the Directional Derivative of $f\ (x,\ y) = 2x^2y^3 + 6xy$ at $(1,\ 1)$ in the direction of a unit vector whose angle with the positive x-axis is π/6.

Solution: Since $\frac{\partial f}{\partial x} = 4xy^3 + 6y$ and $\frac{\partial f}{\partial y} = 6x^2y^2 + 6x$, we have

$\nabla f\ (x,\ y) = (4xy^3 + 6y)\ \boldsymbol{i} + (6x^2y^2 + 6x)\ \boldsymbol{j}$ and

$\nabla f\,(1,1) = 10\ \boldsymbol{i} + 12\ \boldsymbol{j}$

Now, at $\theta = \frac{\pi}{6}$, $u = cos\theta \cdot \boldsymbol{i} + sin\theta \cdot \boldsymbol{j}$

$$u = \frac{\sqrt{3}}{2}\ \boldsymbol{i} + \frac{1}{2}\ \boldsymbol{j}$$

Therefore, $D_u f(1,\ 1) = f\ (1,\ 1) \cdot u$

$$= (10\boldsymbol{i} + 12\boldsymbol{j}) \cdot \left(\frac{\sqrt{3}}{2}\ \boldsymbol{i} + \frac{1}{2}\ \boldsymbol{j}\right)$$

$$= 5\sqrt{3} + 6.$$

Example 2: Find the Directional Derivative of $F\ (x,\ y,\ z) = xy^2 - 4x^2\ y + z^2$ at $(1,\ 1,\ 2)$ in the direction of $6\ \boldsymbol{i} + 2\ \boldsymbol{j} + 3\ \boldsymbol{k}$.

Solution: We have $\frac{\partial F}{\partial x} = y^2 - 8xy$, $\frac{\partial F}{\partial y} = 2xy - 4x^2$, and $\frac{\partial F}{\partial z} = 2z$

so that $\nabla F\ (x,\ y,\ z) = (y^2 - 8xy)\boldsymbol{i} + (2xy - 4x^2)\boldsymbol{j} + 2z\ \boldsymbol{k}$

$\nabla F\ (1,\ 1,\ 2) = 7\ \boldsymbol{i} - 2\ \boldsymbol{j} + 4\ \boldsymbol{k}$.

Since $|6\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}| = 7$

then $\boldsymbol{u} = \frac{6}{7}\boldsymbol{i}+\frac{2}{7}\boldsymbol{j}+\frac{3}{7}\boldsymbol{k}$ is a unit vector in the indicated direction.

Therefore,

$$D_uF(1,\ 1,\ 2) = \nabla F\ (1,\ 1,\ 2)\cdot \boldsymbol{u}$$

$$= (-7\boldsymbol{i} - 2\boldsymbol{j} + 4\,\boldsymbol{k})\cdot\left(\frac{6}{7}\boldsymbol{i}+\frac{2}{7}\boldsymbol{j}+\frac{3}{7}\boldsymbol{k}\right)$$

$$= \frac{-34}{7}$$

Example 3: In a laboratory studying heat transfer phenomena, a special material exhibits a temperature distribution described by the function $T(x,\ y,\ z) = e^{-(x^2+y^2+z^2)}$. Consider a point (1, 1, 1) located within this material. Calculate the rate of heat flow from $(1,1,1)$ in the direction specified by the vector $v = \langle 1,\ -1,\ 2\rangle$. What insights can be gained from the numerical value of the directional derivative in terms of heat transfer at the given point?

Solution: $\nabla T\ (x,\ y,\ z) = -2\ e^{-(x^2+y^2+z^2)}(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k})$.

$\nabla T\ (1,\ 1,\ 1) = -2\ e^{-3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$.

$\boldsymbol{v} = \frac{1}{\sqrt{6}}\boldsymbol{i} - \frac{1}{\sqrt{6}}\boldsymbol{j} + \frac{2}{\sqrt{6}}\boldsymbol{k}$ is a unit vector in the indicated direction.

Therefore,

$$D_uT(1,\ 1,\ 1) = \nabla T\ (1,\ 1,\ 1)\cdot \boldsymbol{v} \quad = -2\ e^{-(x^2+y^2+z^2)}(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k})\cdot\left(\frac{1}{\sqrt{6}}\boldsymbol{i} - \frac{1}{\sqrt{6}}\boldsymbol{j} + \frac{2}{\sqrt{6}}\boldsymbol{k}\right)$$

$= \frac{-4\ e^{-3}}{\sqrt{6}}$ ⇒Temperature is decreasing in the specified direction from the given point.

Example 4: Consider a composite material with two layers. The upper layer has a stress distribution given by $\sigma_1(x,\ y,\ z) = 5x + 2y + 3z$, MPa and the lower layer has a stress distribution given by $\sigma_2(x,\ y,\ z) = 8x + 12y - 4z$, MPa. Obtain the angle between the surfaces at the given points.

Solution: $\nabla\sigma_1 = 5\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k}$ and $\nabla\sigma_2 = 8\,\boldsymbol{i} + 12\,\boldsymbol{j} - 4\,\boldsymbol{k}$

$|\nabla\sigma_1| = \sqrt{38}$ and $|\nabla\sigma_2| = \sqrt{224}$

$$\boldsymbol{cos\theta} = \frac{\nabla\sigma_1\cdot\nabla\sigma_2}{|\nabla\sigma_1||\nabla\sigma_2|}$$

$$\boldsymbol{cos\theta} = \frac{(5\,\boldsymbol{i} + 2\,\boldsymbol{j} + 3\,\boldsymbol{k})\cdot(8\,\boldsymbol{i} + 12\,\boldsymbol{j} - 4\,\boldsymbol{k})}{\sqrt{38}\cdot\sqrt{224}}$$

$$= \frac{52}{\sqrt{38}\cdot\sqrt{224}}$$

$$\boldsymbol{\theta = cos^{-1}}\left(\frac{52}{\sqrt{38}\cdot\sqrt{224}}\right)$$

Applications of Vector Differentiation:

1. **Climate Modeling:** Simulating and analyzing vector fields representing atmospheric circulation and ocean currents.
2. **Geographical Information Systems (GIS):** Studying spatial patterns and Gradients in environmental data.

Physics and Engineering:

1. Applied in mechanics, electromagnetism, and fluid dynamics to describe and analyse vector fields.
2. Used in computer graphics, computer vision, and simulations involving vector quantities.

Economics and Finance:

1. Utilized in optimization problems, like maximizing profit or minimizing cost functions in economics.

Practice Problems:

1. For a temperature distribution function $\boldsymbol{T(x,y,z) = x^2z + y^2x + z^2y}$ find the direction of the steepest increase in temperature at (1, 2, 3). Ans:$10\,\boldsymbol{i} + 13\,\boldsymbol{j} + 13\,\boldsymbol{k}$
2. Consider an electric circuit where the electric potential is described by the function $\boldsymbol{V(x,\ y) = 5x^2 + 3y^2}$. Calculate directional derivative of the electric potential at the point (2, 3) when it moves in a specific direction $u = \boldsymbol{-i + j}$. Ans: $D_uV(2,\ 3) = |-\sqrt{2}|$
3. Find the Directional Derivative of $\varphi = \boldsymbol{x^2yz} + \boldsymbol{4xz^2}$ at the point (1, -2, -1) along the vector 2i-j-2k. Ans:$\frac{37}{3}$
4. Find the unit normal to the surface $\boldsymbol{x^2yz} + \boldsymbol{xy^2z} + \boldsymbol{xyz^2} = \boldsymbol{3}$ at (1,1,1). Ans: $\frac{4(i+j+k)}{4\sqrt{3}}$

Home Work Problems:

1. Given a mechanical energy function $\boldsymbol{E(x,\ y) = 2x^2y - 3x^2y^2}$ find the directional derivative at the point (1, -1) used in optimizing mechanical systems in the direction of $u = -2I + J$.
2. A vector field on $\mathbb{R}^2$ is defined by $F(x,y) = -\frac{1}{2}\boldsymbol{i} + (y - x)\boldsymbol{j}$. Describe F by sketching some of the vectors $F(x,\ y)$.

3. In the study of flow patterns in pipes, for the velocity field $V(x, y, z) = 2x^2\boldsymbol{i}+3y^2\boldsymbol{j} - 4z^2\boldsymbol{k}$ find the direction of maximum increase in fluid velocity.

4. To optimize electric field configurations for an electric potential function $V(x, y, z) = x^2-2y^2 + 5zy$ find the direction of the electric field.

5. Find the Directional Derivative of $\boldsymbol{xy} + \boldsymbol{yz} + \boldsymbol{zx}$ at (3, 5, -4) in the direction of the tangent to the curve $\vec{r} = (t^2 - 1)\,\hat{\imath} + (4t - 3)\,\hat{\jmath} + (2t^2 - 6t)\hat{k}$

6. Find the angle between the surface $\boldsymbol{x^2yz} + \boldsymbol{3xz^2} = \boldsymbol{5}$ and $\boldsymbol{x^2yz^3} = \boldsymbol{2}$ at (1, -2, -1).