MODULE – 4
CENTROID OF PLANE FIGURES

Centroid of Plane Figures

Introduction and Applications of Centroid and Centre of gravity. Derivation of Centroid of Rectangle, Triangle, Semicircle and Quarter circle using method of integration including numerical problems on Centroid of composite figures (1st Quadrant and without any deductable areas)

Introduction to Centroid:

The centroid of a 2D plane figure is the point at which the entire area of the figure could be balanced. It's the geometric centre or average position of all the individual points that make up the shape. In the context of civil engineering, centroids play a critical role in analysing structures and materials for their stability and distribution of forces

In engineering applications, particularly, calculating the centroid is essential for comprehending how forces and loads are distributed in a building. Whether it's a massive bridge, a towering skyscraper, or small beam, understanding the centroid concept lies a crucial role for ensuring that the structures remain steady and reliable under the applied forces. By pinpointing the centroid, engineers can make decisions about, load-bearing capacities, and overall structural integrity of a building.

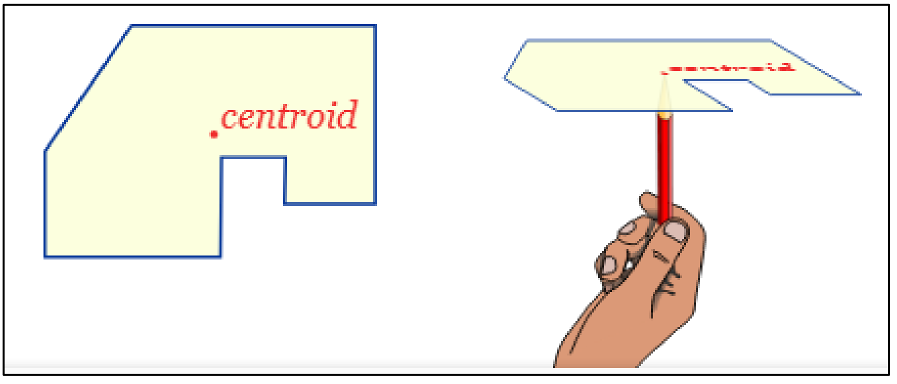
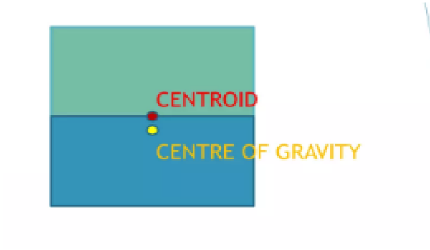


Figure 1: Centroid of Irregular Plane figures



Applications of Centroid Plane Figures

a) **Structural Analysis**: In designing bridges, buildings, and other structures, determining the centroid is crucial for understanding load distribution, calculating moments, and ensuring balanced support.

b) **Hydrology**: In hydrological studies, the centroid of a watershed can help determine the average runoff direction and velocity.

c) **Soil Mechanics**: In geotechnical engineering, centroids are used to analyze pressure distribution and stability of soil masses.

d) **Transportation Engineering**: In road design, centroids of cross-sections are used to ensure proper drainage and balance of the roadway.

Centre of Gravity in Civil Engineering:

The center of gravity (CG) of an object is the point through which the entire weight of the object acts due to the force of gravity. In civil engineering, the concept is significant for understanding stability, equilibrium, and overall behaviour of structures.

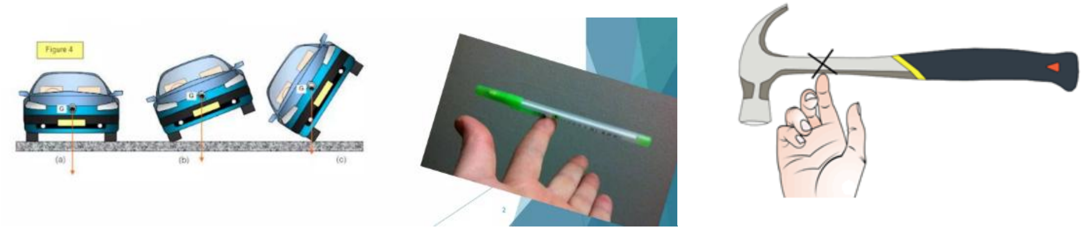


Figure 3: Centre of gravity

Applications of Centre of Gravity

a) Structural Stability: Engineers assess the centre of gravity of structures to ensure that they remain stable and do not topple under various loads.

b) Slope Stability: Determining the centre of gravity is vital in analysing the stability of slopes and preventing landslides.
c) Retaining Wall Design: Centre of gravity is considered in retaining wall designs to prevent tipping and maintain stability.
d) Load Balancing: Engineers consider centre of gravity in the design of equipment and machinery to prevent tipping during operation or transport.
e) Structural Components: For columns, beams, and other elements, understanding the centre of gravity is crucial for load distribution and equilibrium analysis

Terminology:

a) Centroid:

Centroid is the point where the whole area of the plane figure is assumed to be concentrated. It is represented as ' G'.

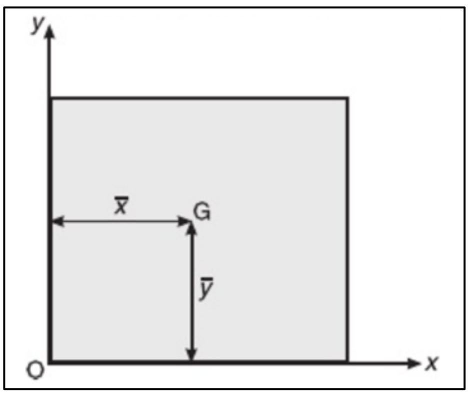


Figure. 4 Centroid of the plane figure.

b) Centre of gravity:

It is the point where the whole weight of the body is assumed to be concentrated. It is the point on which the body can be balanced. It is the point through which the weight of the body is assumed to act. This point is usually denoted by 'C.G.' or 'G'.

c) Centroidal axis:

The axis which passes through the centroid of the given figure is known as centroidal axis, such as the axis X -X and the axis Y -Y shown in Figure 5

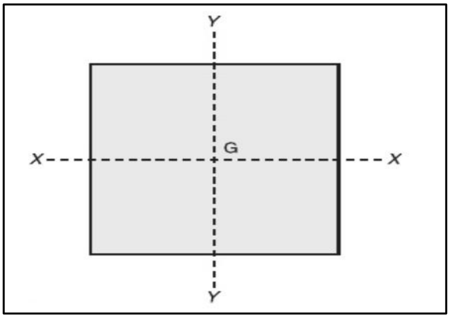


Figure.5 Centroidal axis

d) Axis of Reference:

These are the axes with respect to which the centroid of a given figure is determined as shown in Figure 6.

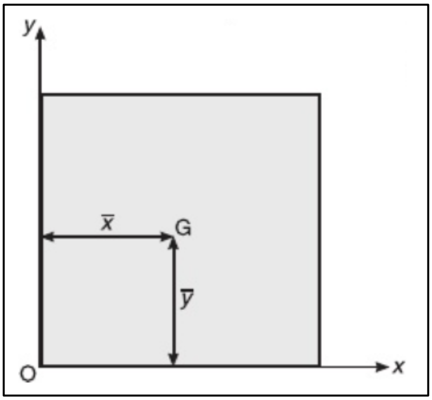


Figure. 6 Axis of Reference

e) **Symmetrical Axis**

It is the axis which divides the whole figure into equal parts, such as the axis $X-X$ and the axis $Y-Y$ shown in Figure 7.

(a) For a figure which is symmetrical about both the axes, $\bar{x}=0$ and $\bar{y}=0$.

(b) For a figure which is symmetrical about the $Y-Y$ axis, $\bar{x}=0$. Such a figure which is symmetrical about the $Y-Y$ axis is shown in Figure 8. The area on the left-side of the $Y-Y$ axis is equal to the area on the right side of the $Y-Y$ axis.

(c) For a figure which is symmetrical about the $X-X$ axis, $\bar{y}=0$. Such a figure which is symmetrical about the $X-X$ axis is shown in Figure 9.

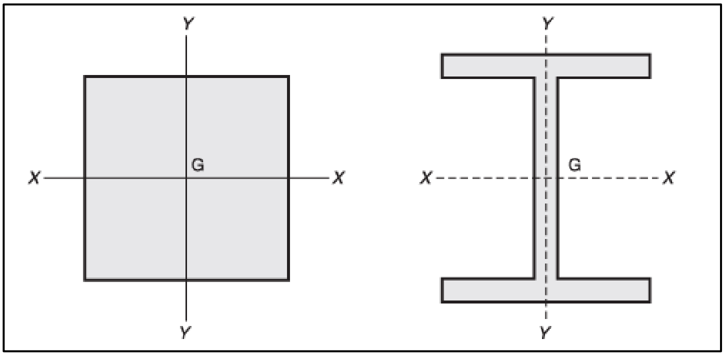


Figure 7: Symmetrical axis

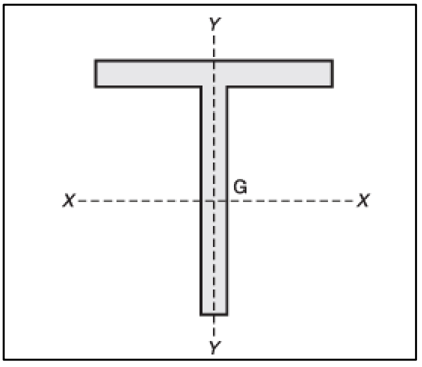


Figure 8: Symmetry about the Y–Y axis.

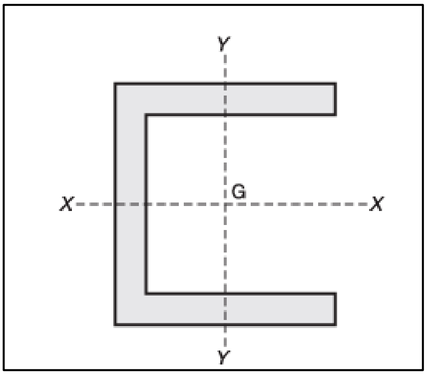


Figure 9: Symmetry about the x–x axis.

For a figure which does not have any axis of symmetry, we calculate both x and y. Such a figure which does not have any axis of symmetry is shown in Figure 10.

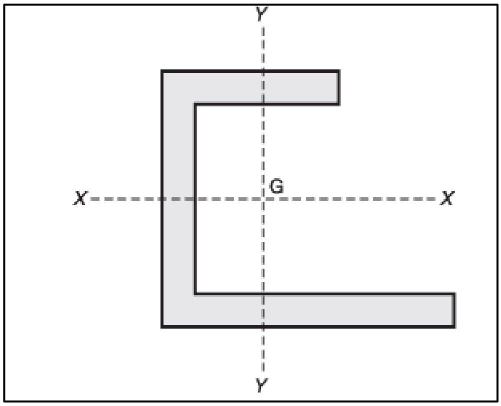


Figure 10: Neither the X–X axis nor the Y–Y axis is the axis of symmetry

Derivation of Centroid of Some Important Geometrical Figures

a) Rectangle

b) Triangle

c) Semicircle

d) Quarter Circle

a) Rectangle:

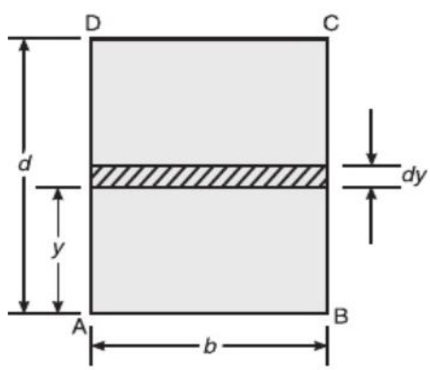


Figure.1.1 Rectangular lamina

Let us consider a rectangular lamina of area b X d as shown in figure 1.1. Now consider a horizontal elementary strip of area *b X dy,* which is at a distance y from the reference axis AB.

Moment of area of elementary strip about AB = *b X dy X y*

Sum of moments of such elementary strips about AB is given by

$$\int_0^d b \times dy \times y$$

$$= b\int_0^d y \cdot dy$$

$$= b \times \left[\frac{y^2}{2}\right]_0^d$$

$$= \frac{bd^2}{2}$$

Moment of total area about AB = $bd \times \bar{y}$

Apply the principle of moments about AB,

$$\frac{bd^2}{2} = bd \times \bar{y} \quad \text{or} \quad \bar{y} = \frac{d}{2}$$

By considering a vertical strip, similarly, we can prove that

$$\bar{x} = \frac{b}{2}$$

b) Triangle: Consider a triangular lamina of area (1/2) x b x d as shown in figure 1.2

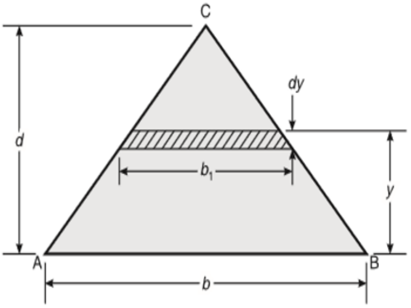


Figure 12. Triangular lamina

Now consider an elementary strip of area $b_1 \times dy$ which is at a distance y from the reference axis AB.

Using the property of similar triangles, we have

$$\frac{b_1}{b} = \frac{d-y}{d}$$

or

$$b_1 = \frac{(d-y)b}{d}$$

$$\text{Area of the elementary strip} = b_1 \times dy = \frac{(d-y)b \cdot dy}{d}$$

Moment of area of elementary strip about AB

$$= \text{area} \times y$$

$$= \frac{(d-y)b \cdot dy \cdot y}{d}$$

$$= \frac{b \cdot dy \cdot d \cdot y}{d} - \frac{by^2 \cdot dy}{d}$$

$$= by \cdot dy - \frac{by^2 \cdot dy}{d}$$

Sum of moments of such elementary strips is given by

$$\int_0^d by \cdot dy - \int_0^d \frac{by^2}{d} \cdot dy$$

$$= b \times \left[\frac{y^2}{2}\right]_0^d - \frac{b}{d}\left[\frac{y^3}{3}\right]_0^d$$

$$= \frac{bd^2}{2} - \frac{bd^3}{3d}$$

$$= \frac{bd^2}{2} - \frac{bd^2}{3}$$

$$= \frac{bd^2}{6}$$

Moment of total area about AB $= \frac{1}{2}bd \times \bar{y}$

Applying the principle of moments,

$$\frac{bd^2}{6} = \frac{1}{2} \times bd \times \bar{y}$$

$$\therefore \quad \bar{y} = \frac{d}{3}$$

c)Semicircle

Consider a semicircular lamina of area $\frac{\pi r^2}{2}$ as shown in Figure 1.3 . Now consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of θ from the x-axis, whose centre of gravity is at a distance of $\frac{2}{3}R$ from O and its projection on the x-axis = $\left(\frac{2}{3}\right)R\cos\theta$.

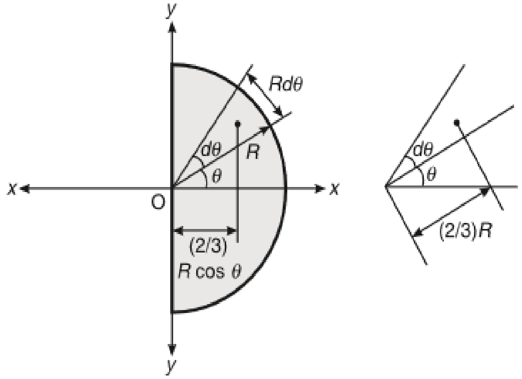


Figure 1.3. Semi-circular lamina

Moment of area of elementary strip about the y-axis $= \frac{1}{2} \times R^2 \cdot d\theta \cdot \left(\frac{2}{3}\right) R \cos\theta$

$$= \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

Sum of moments of such elementary strips about the y-axis

$$= \int_{-\pi/2}^{\pi/2} \frac{R^3}{3} \cos\theta \cdot d\theta$$

$$= \frac{R^3}{3} [\sin\theta]_{-\pi/2}^{\pi/2}$$

$$= \frac{R^3}{3}\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right] = \frac{2R^3}{3}$$

Moment of total area about the y-axis

$$= \frac{\pi R^2}{2} \times \bar{x}$$

Using the principle of moments

$$\frac{2R^3}{3} = \frac{\pi R^2}{2} \times \bar{x}$$

$\therefore$
$$\bar{x} = \frac{2R^3 \times 2}{3R^2 \pi}$$

or
$$\bar{x} = \frac{4R}{3\pi}$$

d) Quarter circle

Consider a quarter circular lamina of area $\frac{\pi R^2}{4}$ as shown in Figure 1.4. Consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of θ from the x-axis, whose centre of gravity is at a distance of $\frac{2}{3} R$ from O and its projection on x-axis $= \frac{2}{3} R \cos\theta$.

Moment of area of elementary strip about the y-axis

$$= \frac{2}{3} R\cos\theta \times \frac{1}{2} \times R^2 \cdot d\theta = \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

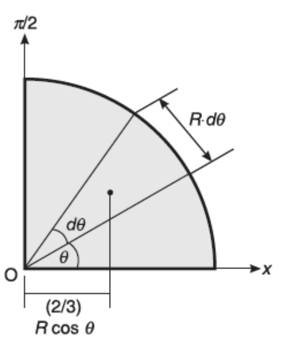


Figure 1.4 Quarter circular lamina.

Sum of moments of such elementary strips about the y-axis

$$= \int_0^{\pi/2} \frac{R^3}{3} \cos\theta \cdot d\theta$$

$$= \frac{R^3}{3}\left[\sin\frac{\pi}{2}\right]$$

$$= \frac{R^3}{3}$$

Moment of total area about the y-axis

$$= \frac{\pi R^2}{4} \times \bar{x}$$

Using the principle of moments,

$$\frac{R^3}{3} = \frac{\pi R^2}{4} \times \bar{x}$$

$\therefore$

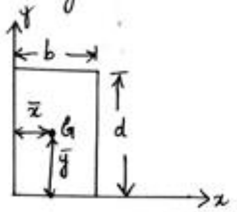
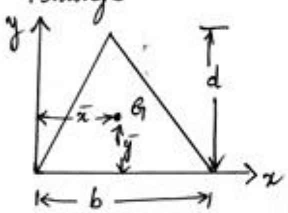
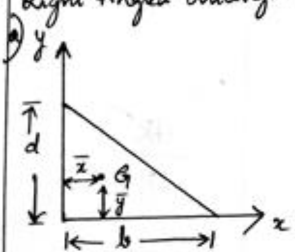
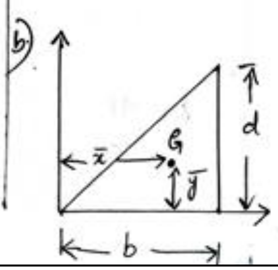
$$\bar{x} = \frac{4R^3 \times 2}{3R^2\pi}$$

or

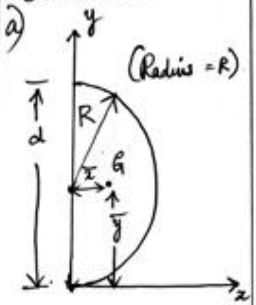
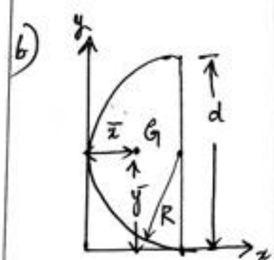
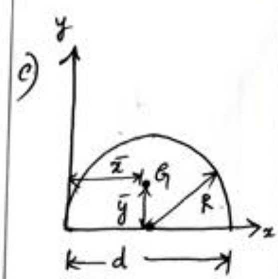
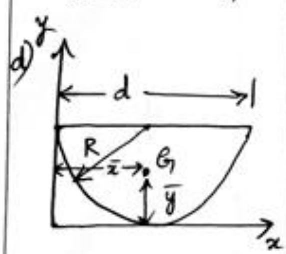
$$\bar{x} = \frac{4R}{3\pi}$$

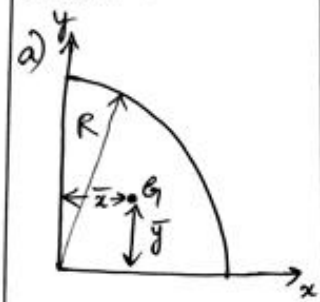
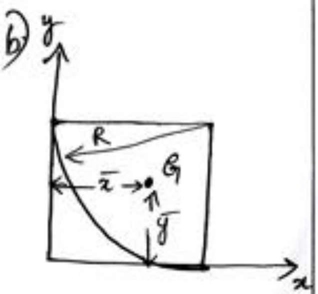
Similarly, we can prove that $\bar{y} = \frac{4R}{3\pi}$.

CENTROID FORMULAS OF SOME IMPORTANT GEOMETRICAL FIGURES

<table>
<tr><th></th><th>$\bar{x}$</th><th>$\bar{y}$</th></tr>
<tr><td>1. Rectangle.

</td><td>$b/2$</td><td>$d/2$</td></tr>
<tr><td>2. Triangle.

</td><td>$b/2$</td><td>$d/3$.</td></tr>
<tr><td>3. Right Angled triangle.
a) 
</td><td>$b/3$</td><td>$d/3$.</td></tr>
<tr><td>b) 
</td><td>$\frac{2b}{3}$.</td><td>$d/3$.</td></tr>
</table>

CENTROID FORMULAS OF SOME IMPORTANT GEOMETRICAL FIGURES

	$\bar{x}$	$\bar{y}$
4. Semi-circle a) 	$\frac{4R}{3\pi}$	$d/2$
b) 	$\left(R - \frac{4R}{3\pi}\right)$	$d/2$
c) 	$d/2$	$\frac{4R}{3\pi}$
d) 	$d/2$	$R - \frac{4R}{3\pi}$

	$\bar{x}$	$\bar{y}$
Quarter Circle a) 	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$.
b) 	$\left(R - \frac{4R}{3\pi}\right)$.	$\left(R - \frac{4R}{3\pi}\right)$

Problems on Centroid
Tips to solve problems

1. The centroid always lies on the symmetrical axis.
2. Identify the symmetrical axes, if any. Chose them as the reference axes. If no symmetrical axis is available, choose the left-hand bottom corner of the given figure

as the origin so that the entire figure lies in the first quadrant (to avoid the negative centroidal values).

3. Sub-divide the given figure into known geometrical shapes and identify their individual centroids by denoting them with C_i where i is the number of the sub-divided shapes.
4. Enter the values in tabular columns as shown below:

Component	Area (a)	Centroidal distance from y-axis (x)	Centroidal distance from x-axis (y)	ax	ay
Sum	Σa			Σax	Σay

5. Indicate the calculations like enter the $x \times y$ values for area and then enter the result (do not enter the result directly). For the centroidal distances, enter the equation with the substituted numerical values like $3 + \frac{4\times5}{3\pi}$ etc. The areas are negative if hollow and the centroidal values are negative if they are below the x-axis or left of the y-axis.
6. Compute the centroidal values as $\bar{x} = \frac{\Sigma ax}{\Sigma a}$ and $\bar{y} = \frac{\Sigma ay}{\Sigma a}$.
7. The centroidal value of a triangle is always $\frac{1}{3}$rd the length from the base and $\frac{2}{3}$rd the height from the apex.
8. The centroidal value of a semicircle is always measured as $\frac{4r}{3\pi}$ from the base of the semicircle (normal to base) or along the symmetrical axis.

Problem 1: Find the centroid of Figure A

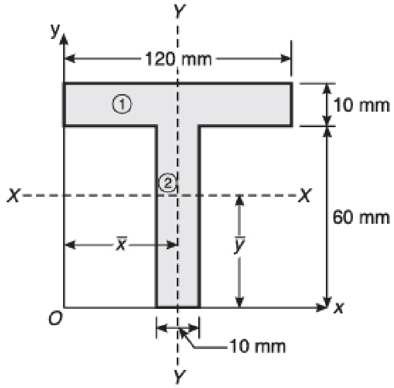
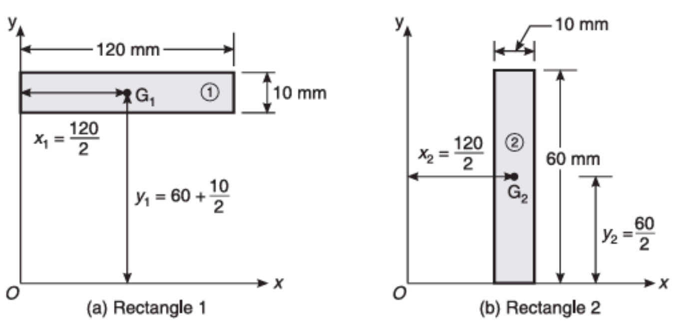


Figure A

Solution

Component	*Area, a(mm²)*	*Centroidal distance from y-axis (x)*	*Centroidal distance from x-axis (y)*	*ax*	*ay*
Rectangle 1	$120 \times 10 = 1200$	$\frac{120}{2} = 60$	$60 + \frac{10}{2} = 65$	72,000	78,000
Rectangle 2	$10 \times 60 = 600$	$\frac{120}{2} = 60$	$\frac{60}{2} = 30$	36,000	18,000
Sum	$\Sigma a = 1800$			108,000	96,000



$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{108,000}{1800} = 60 \text{ mm}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{96,000}{1800} = 53.33 \text{ m}$$

Ans.

The given figure is symmetrical about the y-axis, so $\bar{x}$ can be directly written as 60 mm.

Problem 2: Find the centroid of Figure B .

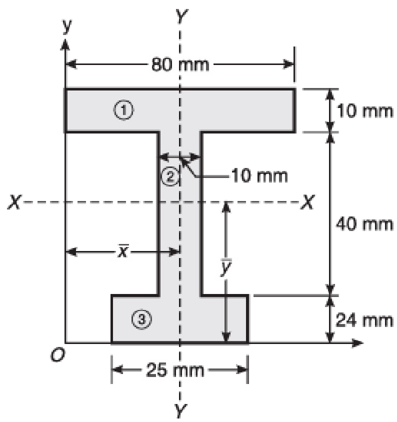


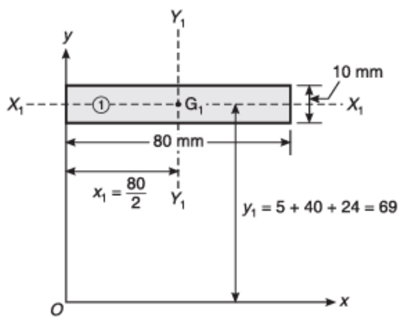
Figure B

The given figure is symmetrical about the $Y–Y$ axis and hence we consider it as the reference y-axis.

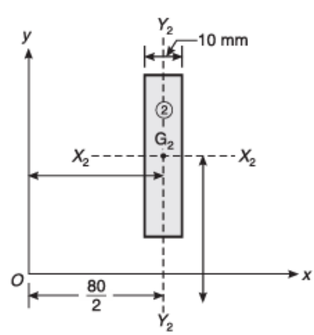
Solution

Component	*Area, a*	*Centroidal distance from the x-axis (y)*	*a y*
Rectangle 1	$10 \times 80 = 800$	$24 + 40 + \frac{10}{2} = 69$	55,200
Rectangle 2	$10 \times 40 = 400$	$24 + \frac{40}{2} = 44$	17,600
Rectangle 3	$25 \times 24 = 600$	$\frac{24}{2} = 12$	7200
Sum	$\Sigma a = 1800$		$\Sigma ay = 80{,}000$

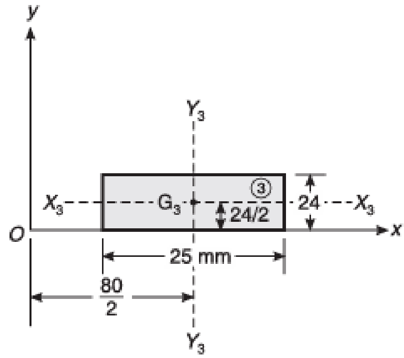
$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{80{,}000}{1800} = 44.44 \text{ mm}; \quad \bar{x} = \frac{80}{2} = 40 \text{ mm} \qquad \textbf{Ans.}$$



(a) Rectangle 1



(b) Rectangle 2



(c) Rectangle 3

Problem 3: Determine the centroid of Figure C

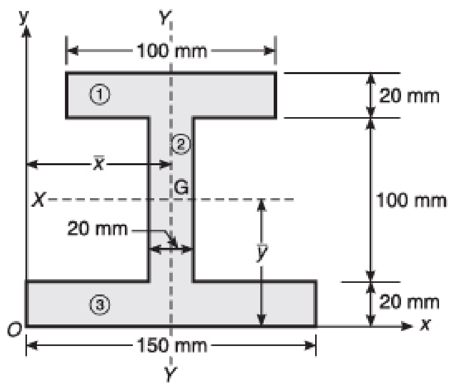


Figure C

Solution

Component	*Area, a*	*y*	*ay*
Rectangle 1	$100 \times 20 = 2000$	$20 + 100 + \frac{20}{2} = 130$	260,000
Rectangle 2	$100 \times 20 = 2000$	$20 + \frac{100}{2} = 70$	140,000
Rectangle 3	$150 \times 20 = 3000$	$\frac{20}{2} = 10$	30,000
Sum	$\Sigma a = 7000$		$\Sigma ay = 430{,}000$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{430{,}000}{7000} = 61.429 \text{ mm}; \quad \bar{x} = \frac{150}{2} = 75 \text{ mm}$$ **Ans.**

Problem 4 : Determine the centroid of Figure D

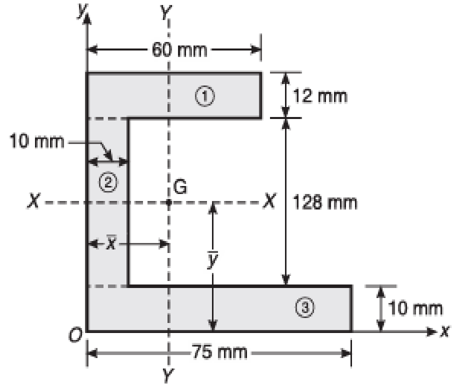


Figure D

Component	*Area, a*	*x*	*y*	*ax*	*ay*
Rectangle 1	$60 \times 12 = 720$	$\frac{60}{2} = 30$	$10 + 128 + \frac{12}{2} = 144$	21,600	103,680
Rectangle 2	$10 \times 128 = 1280$	$\frac{10}{2} = 5$	$10 + \frac{128}{2} = 74$	6400	94,720
Rectangle 3	$10 \times 75 = 750$	$\frac{75}{2} = 37.5$	$\frac{10}{2} = 5$	28,125	3750
Sum	$\Sigma a = 2750$			$\Sigma ax = 56,125$	$\Sigma ay = 202,150$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{56,125}{2750} = 20.409 \text{ mm}; \quad \bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{202,150}{2750} = 73.509 \text{ mm} \quad \textbf{Ans.}$$

Problem 5: Determine the centroid of Figure E

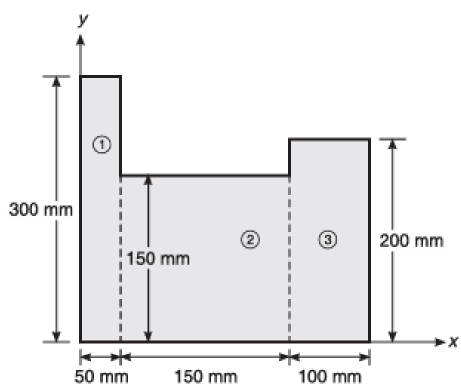


Figure E

Solution

Component	*Area, a*	*x*	*y*	*ax*	*ay*
Rectangle 1	50 × 300 = 15,000	$\frac{50}{2} = 25$	$\frac{300}{2} = 150$	375,000	2,250,000
Rectangle 2	150 × 150 = 22,500	$50 + \frac{150}{2} = 125$	$\frac{150}{2} = 75$	2,812,500	1,687,500
Rectangle 3	200 × 100 = 20,000	$200 + \frac{100}{2} = 250$	$\frac{200}{2} = 100$	5,000,000	2,000,000
Sum	Σa = 57,500			Σax = 8,187,500	Σay = 5,937,500

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{8,187,500}{57,500} \; 142.391 \text{ mm}; \quad \bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{5,937,500}{57,500} = 103.261 \text{ mm} \quad \textbf{Ans.}$$

Problem 6: Determine the centroid of Figure F

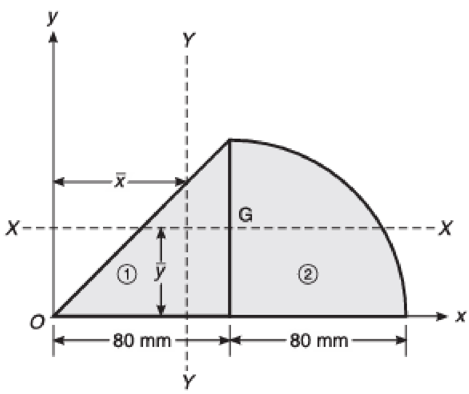


Figure F

Solution

Component	*Area, a*	*x*	*y*	*ax*	*ay*
Triangle 1	$\frac{1}{2} \times 80 \times 80$ = 3200	$\frac{2}{3} \times 80$ = 53.33	$\frac{1}{3} \times 80$ = 26.67	170,665.6	85,334.4
Quarter circle 2	$\frac{\pi \times 80^2}{4}$ = 5026.548	$80 + \frac{4 \times 80}{3 \times \pi}$ = 113.953	$\frac{4 \times 80}{3 \times \pi}$ = 33.953	572,790.224	170,666.384
Sum	Σa = 8226.548			Σax = 743,455.82	Σay = 256,000.78

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{743,455.824}{8226.548} = 90.373 \text{ mm}; \quad \bar{y} = \frac{\Sigma ay}{a} = \frac{256,000.784}{8226.548} = 31.119 \text{ mm} \quad \textbf{Ans.}$$

Problem 7: Determine the centroid of Figure G

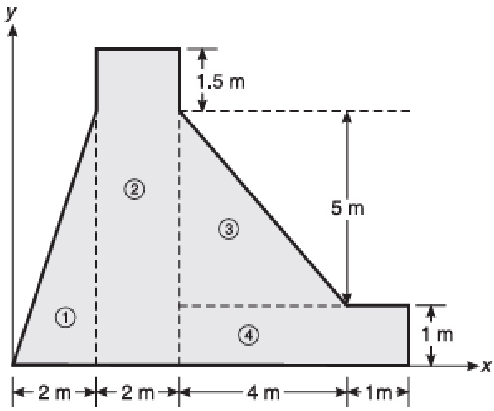


Figure G

Solution

Component	*Area, a*	*x*	*y*	*ax*	*ay*
Triangle 1	$\frac{1}{2}\times 6\times 2 = 6$	$\frac{2}{3}\times 2 = 1.33$	$\frac{1}{3}\times 6 = 2$	7.98	12
Rectangle 2	$2\times 7.5 = 15$	$\frac{2}{2}+2 = 3$	$\frac{7.5}{2} = 3.75$	45	56.25
Triangle 3	$\frac{1}{2}\times 4\times 5 = 10$	$4+\frac{1}{3}\times 4 = 5.33$	$1+\frac{1}{3}\times 5 = 2.67$	53.33	26.67
Rectangle 4	$5\times 1 = 5$	$\frac{5}{2}+4 = 6.5$	$\frac{1}{2} = 0.5$	32.5	2.5
Sum	$\Sigma a = 36$			$\Sigma ax = 138.81$	$\Sigma ay = 97.42$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{138.81}{36} = 3.856 \text{ m}$$

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{97.42}{36} = 2.706 \text{ m}$$

Ans.

Problem 8

Determine the centroid of the plane shown in figure H with respect to base.

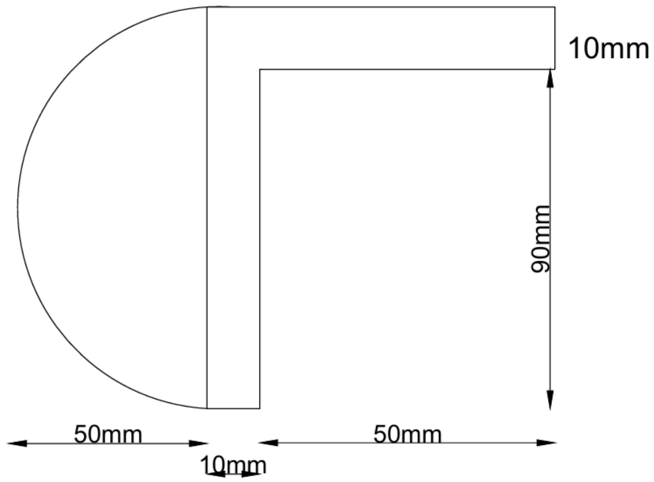
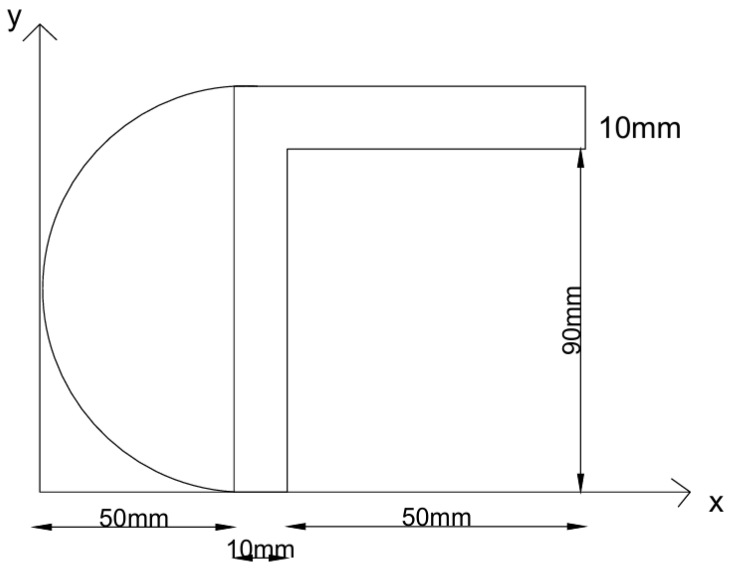


Figure H

Solution



Component	Area (mm^2)	X(mm)	Y(mm)	ax	ay
Rectangle 1	(60) (10)	80	95	48000	57000
Rectangle 2	(10) (90)	55	45	49500	40500
Semicircle	$\frac{\pi \times 50^2}{2}$	$(50) - \frac{4\times 50}{3\pi}$	150	113016.21	589048.63

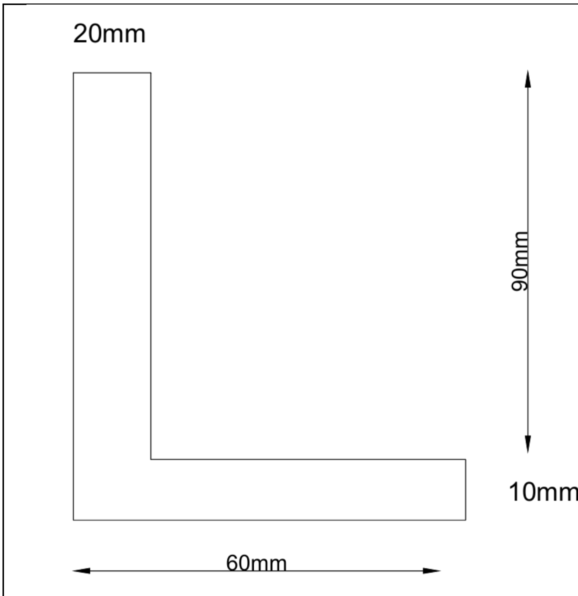
$$\sum A = 5426.99$$
$$\sum Ax = 210516.2$$
$$\sum Ay = 293849.54$$

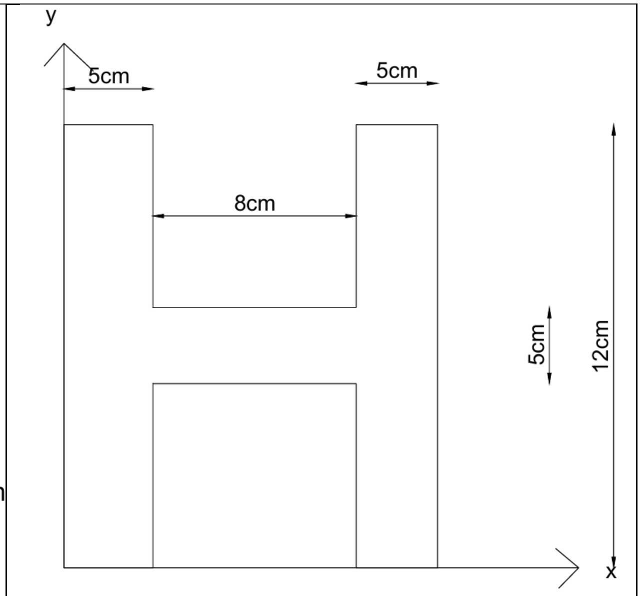
$$\bar{X} = \frac{\sum Ax}{\sum A}$$
$$= \frac{210516.2}{5426.99}$$
$$\bar{X} = \mathbf{38.79\ mm}$$

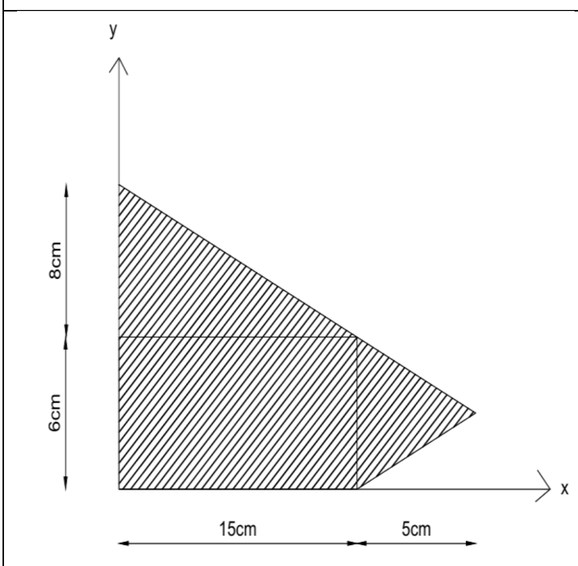
$$\bar{Y} = \frac{\sum Ay}{\sum A}$$
$$= \frac{293849.54}{5426.99}$$
$$\bar{Y} = \mathbf{54.15\ mm}$$

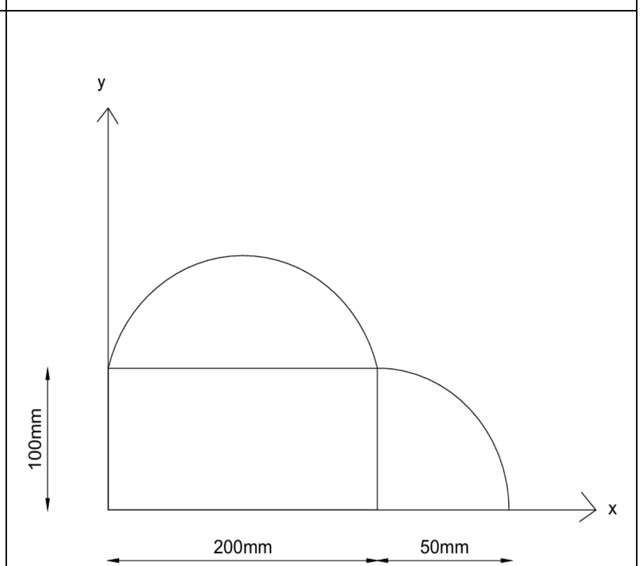
Exercise problems

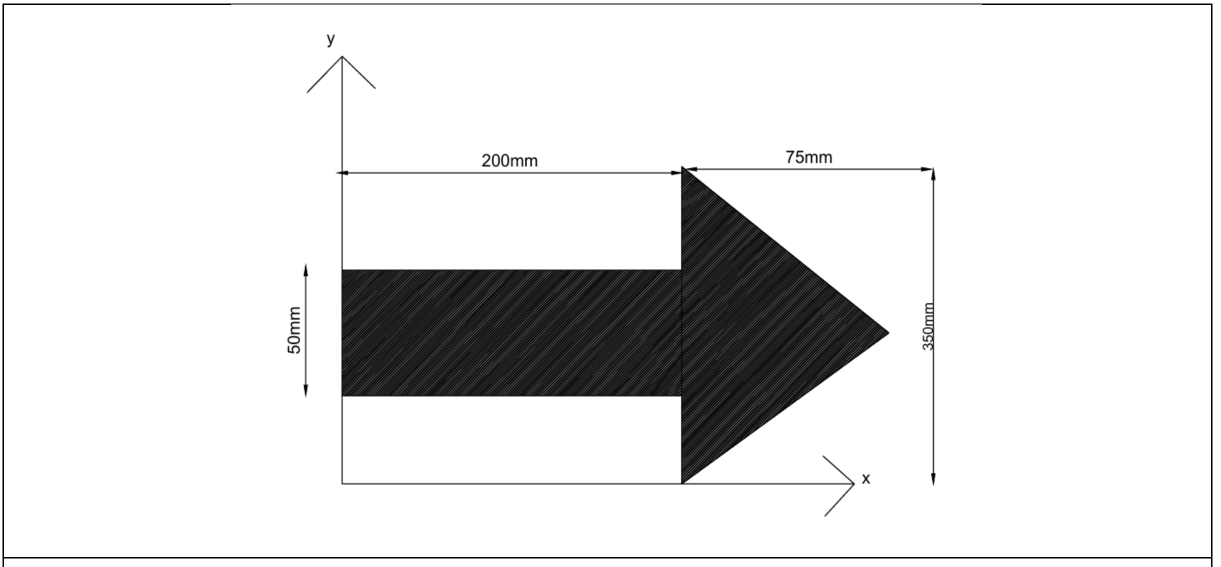
Determine the centroid for the following figures

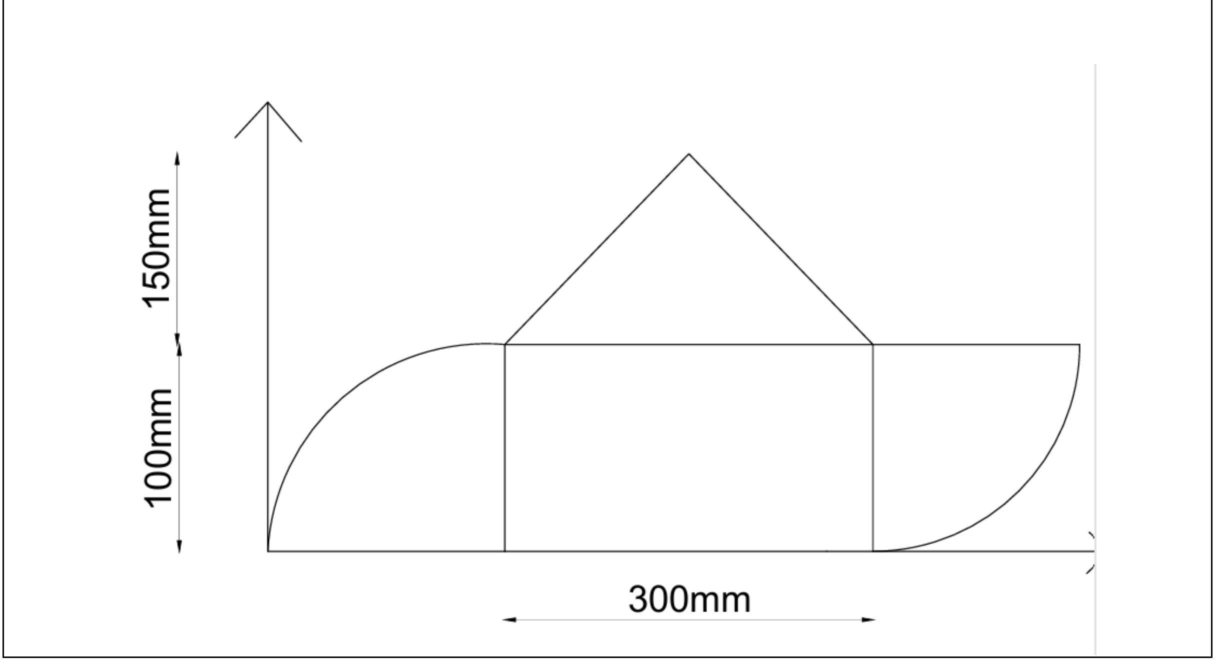









y
200mm
75mm
50mm
350mm
x


150mm
100mm
300mm