Module 5
MOMENT OF INERTIA OF PLANE FIGURES

Introduction to Inertia and Moment of Inertia, Applications, Parallel axes theorem, Perpendicular axes theorem. Derivation of Moment of inertia of rectangle, triangle, circle, semi-circle and quarter circular areas by the method of integration. Numerical symmetrical Channel Section. problems on Moment of Inertia of T, I and symmetrical Channel Section

Introduction

Newton's First Law of Motion states that a body continues to remain at rest or in uniform motion in a straight line unless compelled by an external force to change that state. This means that a body offers resistance to any change in its state of motion. This property of matter, of resisting any change in its state of rest or uniform motion, is known as inertia.

- For bodies in linear motion, the force required to change their state of motion depends on the mass of the body.
- For bodies in rotation about an axis, the force required to change their state of motion depends on both the mass of the body and its distance from the axis around which it is rotating.

Moment of Inertia (MOI) in Civil Engineering:
Moment of Inertia of a section is a measure of its resistance to bending or deflection under applied loads. It depends on the shape, size, and distribution of material about an axis.

Applications in Civil Engineering

- Determines bending stress in beams under loads.
- Used in design of beams, slabs, and columns to resist bending.
- Helps calculate deflection of beams and cantilevers.
- Important in structural analysis of bridges and frames.
- Used in shaft and column design for torsion and rotation resistance.

Terminology:

a. Moment of Inertia

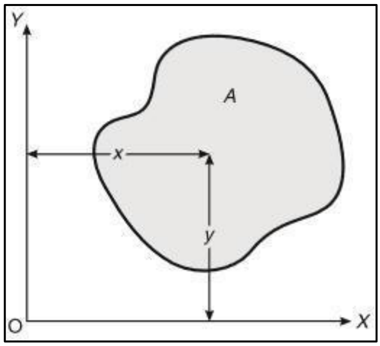


Figure 1. Irregular plane of lamina

Moment of area about the y-axis = first moment of area.

If the first moment of area is multiplied by the perpendicular distance x, it gives Ax^2 known as the second moment of area or moment of inertia.

b. Radius of Gyration

It is the distance from the given axis where the whole area of a plane figure is assumed to be concentrated so as not to alter the moment of inertia about the given axis. It is denoted as k.

$$k = \sqrt{(I / A)}$$

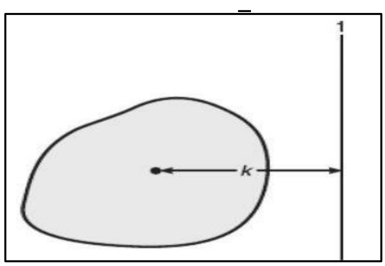


Figure 2. Radius of gyration k

c.Polar moment of inertia

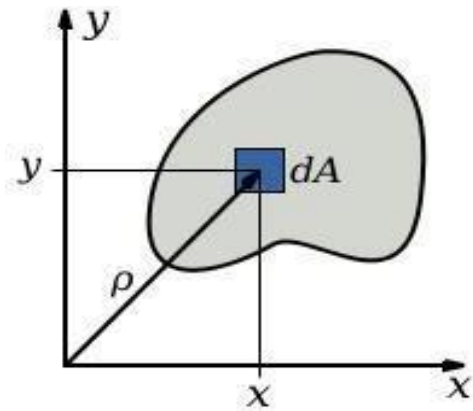


Figure 3. Polar moment of inertia

Polar moment of inertia is calculated for an arbitrary shape about an axis O
Where ρ is the radial distance to the element dA

The polar moment of inertia, also called the second polar moment of area, measures how much a cylindrical object (like a shaft or pipe) resists twisting (torsional deformation). It is used for objects with the same cross-section shape and no bending or warping out of their plane

Polar moment of inertia can be described as the summation of x and y planar moments of inertia, Ix and Iy

$$I_Z = I_x + I_y$$

Least Moment of Inertia (I_l)

Definition: It is the minimum resistance offered by a plane figure to bending, occurring about the axis where the section is weakest.

Explanation:

- The weak axis is the axis along which the material is closer to the axis, so the section cannot resist bending effectively.

- For example, in a rectangular beam, bending about the horizontal axis (width-wise) gives the least MOI, because most of the material is close to this axis.

- This axis is important in structural design, as members must be oriented to avoid bending about the weak axis.

Greatest Moment of Inertia (I_{max})

Definition: It is the maximum resistance offered by a plane figure to bending, occurring about the axis where the section is strongest.

Explanation:

- The strong axis is usually the centroidal axis passing through the centroid and along the longest dimension of the section.
- For a rectangle, bending about the vertical axis (height-wise) gives the greatest MOI, because the material is distributed far from the axis, offering maximum resistance to bending.
- Design engineers orient beams along this axis to maximize stiffness and minimize deflection

Derivation of Moment of Inertia of Some Important Geometrical Figures

a) Rectangle

Let us consider a rectangular lamina of breadth b and depth d whose moment of inertia is to be determined (Figure 4). Now consider an elementary strip of area b. dy at a distance y from the centroidal x–x axis. The moment of inertia of the strip about the x–x axis = b dy y^2. Moment of inertia of the whole figure about the x–x axis

$$= \int_{-d/2}^{d/2} b \cdot dy \times y^2$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$= \frac{b \times d^3}{12}$$

$$\bar{I}_x = \frac{bd^3}{12}$$

$$\bar{I}_y = \frac{db^3}{12}$$

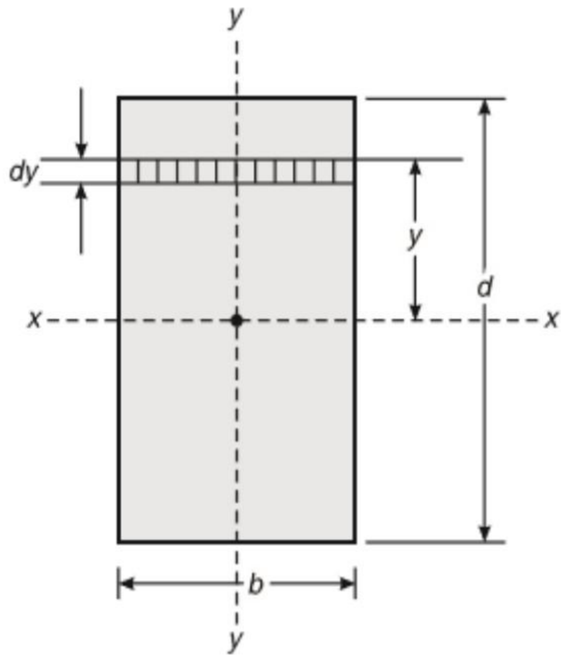


Figure 4.Rectangular lamina

b) Triangle

Let us consider a triangular lamina of base b and depth d as shown in Figure 5. Let us consider an elementary strip of area b_1dy which is at a distance y from base AB. Using the property of similar triangles,

Using the property of similar triangles,

$$\frac{b_1}{b} = \frac{d-y}{d}$$

$$b_1 = \frac{(d-y)b}{d}$$

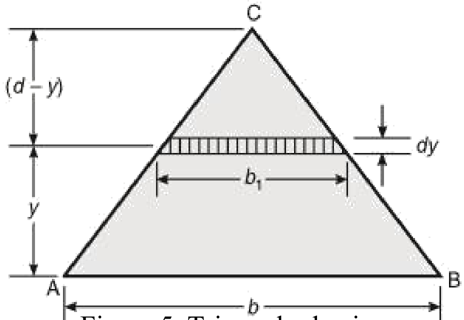


Figure 5. Triangular lamina

Area of the strip $= \dfrac{(d-y)b}{d} \cdot dy$

Moment of inertia of the strip about AB $= \dfrac{(d-y)b}{d} dy \times y^2$

$$= \frac{bdy^2 \cdot dy}{d} - \frac{by^3 \cdot dy}{d}$$

$$= by^2 \cdot dy - \frac{by^3 \cdot dy}{d}$$

Moment of inertia of the whole area about AB,

$$I_{AB} = \int_0^d by^2 dy - \int_0^d \frac{b}{d} y^3 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^d - \frac{b}{d}\left[\frac{y^4}{4}\right]_0^d$$

$$= \frac{bd^3}{3} - \frac{b}{d}\frac{d^4}{4}$$

$$= \frac{bd^3}{3} - \frac{bd^3}{4}$$

$$I_{AB} = \frac{bd^3}{12}$$

Moment of inertia about x–x axis is given by

$$I_{AB} = \overline{I}_x + Ay^2$$

$$\overline{I}_x = I_{AB} - Ay^2$$

$$= \frac{bd^3}{12} - \frac{1}{2}bd\left(\frac{1}{3}d\right)^2 = \frac{bd^3}{36}$$

Therefore, the moment of inertia of the triangle about the centroidal y-axis $= \dfrac{db^3}{36}$.

Parallel axis theorem.

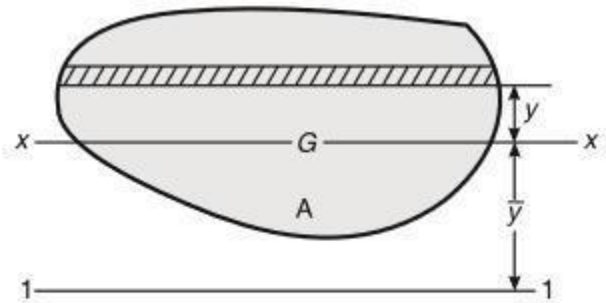


Figure 6: Illustration of parallel axis theorem for moment of inertia about an axis parallel to x–x axis.

Statement: This theorem states that the moment of inertia of plane figure about an axis 1-1, parallel to the centroidal axis, I_x is equal to sum of moment of inertia about centroidal axis, i.e. I_x and the product of area of the plane figure and square of the distance between the two axes.

Proof: Let us consider a plane figure of total area A as shown in Figure 6. Let Ix be the moment of inertia about the x-axis and I 1-1 be the moment of inertia as shown in Figure 6. Let Ix be the moment of inertia about the x-axis and I1–1 the moment of inertia about 1–1 axis.

Let us choose an elemental strip of area da at a distance y from the centroidal axis. Moment of inertia of the strip about x–x axis = $da \cdot y^2$

Moment of inertia of the total area about the x–x axis = Ix =

$\Sigma da \cdot y^2$ Moment of inertia of the strip about 1–1 axis = $da(y + y)^2$

Moment of inertia of the total area about 1–1 axis I1–1 =

$\Sigma da(y^2 + y^2 + 2y\ y)$

$I_{1-1} = \Sigma da y^2 + \Sigma da\ y^2 + 2\ y(\Sigma da y)$

As the distance of C.G. of whole area from the centroidal axis = 0, i.e.

y = 0, we get $\mathbf{I_{1-1} = I_x + A\ y^2}$

Similarly, the moment of inertia about an axis I2–2 as shown in Figure 6 is given

by $\mathbf{I_{2-2} = I_y + Ax^2}$

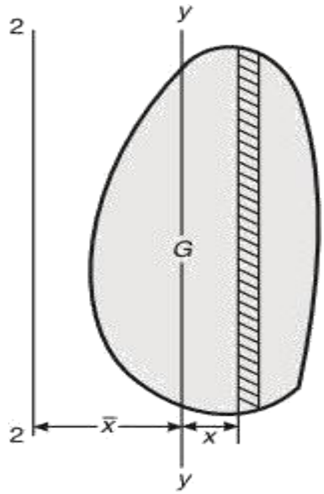


Figure 7: Illustration of parallel axis theorem for moment of inertia about an axis parallel to y – y axis.

Perpendicular axis theorem.

Statement: This theorem states that the moment of inertia of a plane figure about an axis which is perpendicular to the plane of the figure is equal to sum of moment of inertia about two mutually perpendicular axes.

Proof:

Let us consider an irregular figure of total area A as shown in Figure 8. Let us choose an elemental strip of area da at a distance x from y-axis, y from x-axis and r from z-axis, respectively. Then, $r^2 = x^2 + y^2$

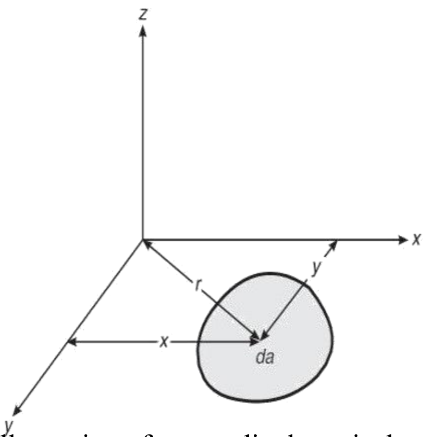


Figure 8. Illustration of perpendicular axis theorem.

Moment of inertia of the strip about x-axis = $da \times y^2$

Moment of inertia of the whole area about the x-axis = $I_x = \Sigma da \cdot y^2$

Similarly, moment of inertia of the strip about y-axis = $da \times x^2$

Moment of inertia of the whole area about y-axis = $I_y = \Sigma da \cdot x^2$

Moment of inertia of the strip about z-axis = $da \times r^2$

Moment of inertia of the whole area about z-axis = $I_Z = \Sigma da \cdot r^2 = \Sigma da(x^2 + y^2)$

$$= \Sigma da \cdot x^2 + \Sigma da \cdot y^2$$

$$\mathbf{I_z = I_y + I_x}$$

PROBLEMS ON MOMENT OF INERTIA

Steps to Solve Any Given Problem

1. Identify the reference axes. If the moment of inertia of the given figure is to be computed about any given axis, then select that axis itself as the reference axis. In general, select the symmetrical axis as the reference axis. If the figure is unsymmetrical, select the left bottom corner of the figure as the origin.
2. Subdivide the compound figure into known geometric shapes and identify the centroids of the sub-figures.
3. Write the tabular format as follows and enter the values.

Reference table

Comp.	*Area(A)*	x	y	Ax	Ay	Ax^2	Ay^2	$\bar{I}_x$	$\bar{I}_y$
1	2	3	4	5	6	7	8	9	10
	ΣA			ΣAx	ΣAy	ΣAx^2	ΣAy^2	ΣI_{gx}	ΣI_{gy}

where

Col. 1 indicates the component or sub-figure number
Col. 2 indicates the area of the sub-figure number
Col. 3 indicates the centroidal distance of the sub-figure from the y-reference axis
Col. 4 indicates the centroidal distance of the sub-figure from the x-reference axis
Col. 5 indicates the product of cells of Col. 2 and Col. 3
Col. 6 indicates the product of cells of Col. 2 and Col. 4
Col. 7 indicates the product of cells of Col. 3 and Col. 5
Col. 8 indicates the product of cells of Col. 4 and Col. 6
Col. 9 indicates the moment of inertia of the sub-figure about its individual centroidal x-axis
Col. 10 indicates the moment of inertia of the sub-figure about its individual centroidal y-axis

4. Compute the centroidal values of the given figure as

$$\bar{x} = \frac{\Sigma Ax}{\Sigma A} \quad \text{and} \quad \bar{y} = \frac{\Sigma Ay}{\Sigma A}$$

5. Compute the moment of inertia of the given compound figure on the x-reference axis using the parallel axis theorem, i.e.

$$I_{1-1} = \bar{I}_x + A\,\bar{y}^2$$

or

$$I_{1-1} = \Sigma \bar{I}_x + \Sigma A y^2$$

6. Compute the moment of inertia of the given compound figure about its centroidal x-axis as

$$\bar{I}_x = I_{1-1} - A\,\bar{y}^2$$

7. Compute the moment of inertia of the given compound figure on the y-reference axis using the parallel axis theorem

$$I_{2-2} = I_x + A\bar{x}^2$$
$$I_{2-2} = \Sigma \bar{I}_x + \Sigma A x^2$$

8. Compute the moment of inertia of the given compound figure about its centroidal y-axis as

$$\bar{I}_y = I_{2-2} - A\bar{x}^2$$

Problem 1 Find the moment of inertia along the horizontal axis and vertical axis passing through the centroid of a section shown in Figure A *VTU (April 2001)*

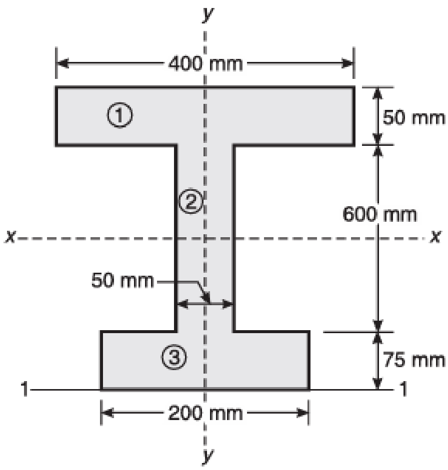


Figure A

Solution The given figure is symmetrical about the y-axis. Therefore, the centroidal y-axis coincides with the reference y-axis. Hence $\bar{x} = 0$.

Moment of inertia about the centroidal x–x axis

$$I_{1-1} = \bar{I}_x + A\,\bar{y}^2 = \Sigma \bar{I}_x + \Sigma A y^2$$

or

$$I_{1-1} - A\,\bar{y}^2 = \bar{I}_x$$
$$I_{2-2} = \bar{I}_y + A\,\bar{x}^2$$

Comp.	*Area* (mm^2)	*y*	*Ay* (mm^3)	*Ay*^{2} (mm^4)	$\overline{I}_x$ (mm^4)	$\overline{I}_y$ (mm^4)
1.	$400 \times 50 = 20{,}000$	$75 + 600 + \frac{50}{2} = 700$	$14{,}000{,}000 = 14 \times 10^6$	9.8×10^9	$\frac{400(50)^3}{12} = 4.167 \times 10^6$	$\frac{50(400)^3}{12} = 266{,}666{,}666.7$
2.	$50 \times 600 = 30{,}000$	$75 + \frac{600}{2} = 375$	$11{,}250{,}000 = 11.25 \times 10^6$	4.219×10^9	$\frac{50 \times 600^3}{12} = 900 \times 10^6$	$\frac{600 \times 50^3}{12} = 6{,}250{,}000$
3.	$200 \times 75 = 15{,}000$	$\frac{75}{2} = 37.5$	562,500	21.09×10^6	$\frac{200 \times 75^3}{12} = 7.03 \times 10^6$	$\frac{75 \times 200^3}{12} = 5{,}00{,}00{,}000$
Σ	65,000		25.812×10^6	1.404×10^{10}	911.197×10^6	322,916,666.7

$$\overline{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{25.812 \times 10^6}{65{,}000} = 397.108 \text{ mm}$$

$$\overline{I}_{1-1} = \overline{I}_x + A\,\overline{y}^2 = \Sigma \overline{I}_x + \Sigma Ay^2 = 1.495 \times 10^{10} \text{ mm}^4$$

$$\overline{I}_x = I_{1-1} - A\,\overline{y}^2 = 1.495 \times 10^{10} - 65000 \times (397.108)^2 = 4.691 \times 10^9 \text{ mm}^4$$ **Ans.**

When the moment of inertia is required on a symmetrical axis, then

$$\overline{I}_y = \Sigma \overline{I}_y = 329{,}216{,}666.7 \text{ mm}^4$$ **Ans.**

Problem 2 Determine the moment of inertia of the unequal I-section about its centroidal axes as shown in Figure B

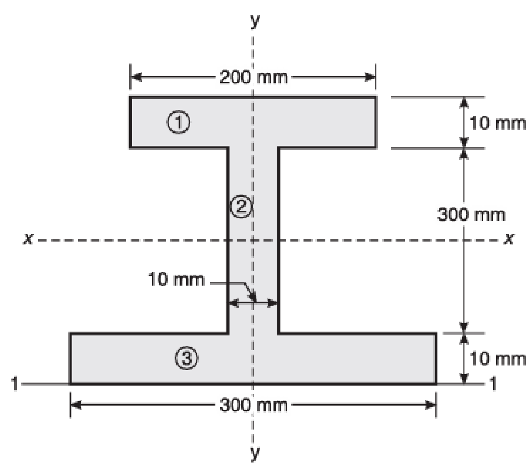


Figure B

Solution

Comp.	*Area* (mm^2)	*y* (mm)	*Ay* (mm^3)	Ay^2 (mm^3)	$\bar{I}_x$ (mm^4)	$\bar{I}_y$ (mm^4)
1.	200×10 = 2000	$10 + 300 + \frac{10}{2}$ = 315	6.3×10^5	1.98×10^8	$\frac{200(10)^3}{12}$ = 16,666.67	$\frac{10(200)^3}{12}$ = 6,666,666.67
2.	300×10 = 3000	$10 + \frac{300}{2}$ = 160	4.8×10^5	0.798×10^8	$\frac{10 \times 300^3}{12}$ = 225,00,000	$\frac{300 \times 10^3}{12}$ = 25,000
3.	300×10 = 3000	$\frac{10}{2} = 5$	0.15×10^5	75×10^3	$\frac{300 \times 10^3}{12}$ = 25,000	$\frac{10 \times 300^3}{12}$ = 2,25,00,000
Σ	8000		11.25×10^5	2.75×10^8	22,541,666.67	29,191,666.7

$$\bar{y} = \frac{\Sigma Ay}{\Sigma a} = \frac{11.25 \times 10^5}{8000} = 140.625 \text{ mm}$$

$$I_{1-1} = \bar{I}_x + A\,\bar{y}^2 = \Sigma \bar{I}_x + \Sigma Ay^2 = 22{,}541{,}666.67 + 2.75 \times 10^8 = 297{,}541{,}666.67 \text{ mm}^4$$

$$\bar{I}_x = I_{1-1} - A\,\bar{y}^2 = 297{,}541{,}666.67 - 8000 \times (140.625)^2 = 139{,}338{,}541.67 \text{ mm}^4$$ **Ans.**

$$\bar{I}_y = 29{,}191{,}666.7 \text{ mm}^4$$ **Ans.**

3. Determine the moment of inertia of the symmetrical channel section shown below about its centroidal axis and calculate Least radius of gyration for the figure 3 shown below.

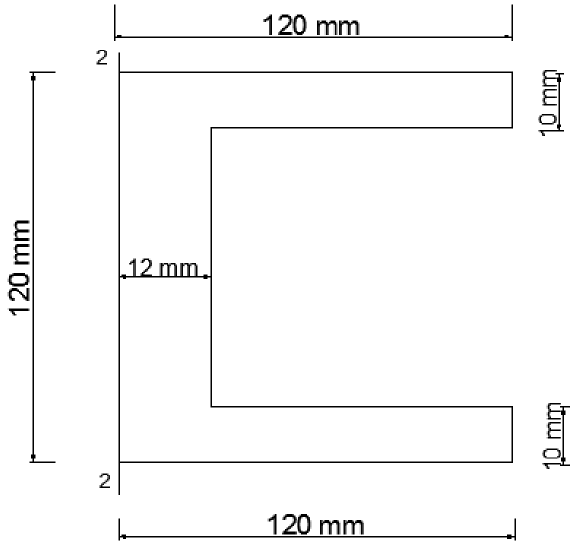


Figure 3

Comp	Area(mm^2)	X(mm)	Ax(mm^3)	Ax^2(mm^4)	Ix(mm^4)	Iy(mm^4)
1	120 x 10 =1200	120/2 =60	1200 x 60 =72000	1200 x 60^2 =4320 x 10^3	$\frac{120 \times 10^3}{12}$ =10000	$\frac{10 \times 120^3}{12}$ =1440000
2	100 x 12 =1200	12/2 =6	1200 x6 =7200	1200 x 6^2 =43200	$\frac{12 x\ 100^3}{12}$ =1000000	$\frac{100 \times 12^3}{12}$ =14400
3	120 x 10 =1200	120/2 =60	1200 x 60 =72000	1200 x 60^2 =4320 x 10^3	$\frac{120 \times 10^3}{12}$ =10000	$\frac{10 \times 120^3}{12}$ =1440000
	∑A= 3600 mm^2		**∑Ax= 151200 mm^3**	**$∑Ax^2$= 8683200 mm^4**	**∑Ix=1020000 mm^4**	**∑Iy=2894400 mm^4**

$\bar{x} = \frac{151200}{3600} = 42$mm, $\bar{Y} = 0$mm.

Moment of inertia about Reference axis $I_{2\text{-}2} = Iy + A\bar{x}^2 = \sum Iy + \sum Ax^2 = 2894400 + 8683200 = 11577600\ mm^4$

Moment of inertia about vertical centroidal axis, $Iy = I_{2\text{-}2} - A\bar{x}^2 = 11577600 - 3600 \times (42)^2 = 5227200\ mm^4$

Moment of inertia about horizontal centroidal axis , $Ix = \sum Ix = 1020000\ mm^4$

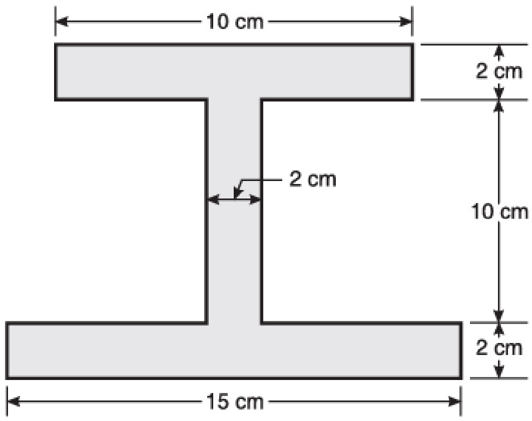
Radius of Gyration $k = \sqrt{\frac{I}{A}}$

$k_x = \sqrt{\frac{Ix}{A}} = \sqrt{\frac{1020000}{3600}}$ =16.83mm ,$k_y = \sqrt{\frac{Iy}{A}} = \sqrt{\frac{5227200}{3600}}$ =38.10mm

Least Radius of Gyration is k_x =16.83mm

Exercise Problems

1.Determine the moment of inertia for the section at the horizontal and vertical axes passing through the centre of gravity of section



2.Determine the moment of inertia of the shaded area for the figure shown below