

MITE

Invent Solutions

NAAC A+ GRADE CGPA: 3.44

MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING

(A Unit of Rajalaxmi Education Trust ®, Mangalore)

Autonomous Institute affiliated to VTU, Belgavi, Approved by AICTE, New Delhi

Accredited by NAAC with A+ Grade & ISO 9001:2015 Certified Institution

MITE

Invent Solutions

MODULE 3 : LINEAR TRANSFORMATIONS

Contents

- Definition and examples
- Matrix representations of a linear transformation
- Change of coordinates and change of basis
- Rank and nullity of a linear operator
- Rank-nullity theorem

MITE

Invent Solutions

Learning Objective

- Understand and articulate the definition of a linear transformation and its properties.
- Understand how matrices represent linear transformations and vice versa.
- Apply the methods for changing coordinates and basis using transition matrices.
- Apply the rank-nullity theorem to relate the dimensions of the domain, range, and kernel.
- Able to find eigenvalues and corresponding eigenvectors for given matrices.
- Able to find dominant eigenvalue and corresponding eigenvector for given matrices.
- Diagonalize a given matrix by finding eigenvectors and constructing the diagonal matrix.

MITE
Invent Solutions

LINEAR TRANSFORMATIONS

Diagnostic (Preview) Questions:

Q1. Fill in the blank: A vector space must be closed under _____ and _____.

Answer: Vector addition and scalar multiplication

Q2. True or False: Every vector space must contain the zero vector.

Answer: True

Q3. Which of the following sets form a vector space under standard operations?

A) All 2×2real matrices

B) All polynomials of degree ≤ 3

C) All vectors in $\mathbb{R}^2$with positive coordinates only

D) The set $\{0\}$

Answer: A, B, D (C fails closure under scalar multiplication with negative scalars)

Q4. True or False: The intersection of two subspaces is always a subspace.

Answer: True

MITE

Invent Solutions

Diagnostic (Preview) Questions:

Q5. Which of the following are subspaces of $\mathbb{R}^2$?

A) $\{(x, y): y = 2x\}$

B) $\{(x, y): y = 2x + 1\}$

C) $\{(x, y): x = 0\}$

D) $\{(x, y): x^2 + y^2 = 1\}$

Answer: A, C (both pass through origin & closed)

LINEAR TRANSFORMATIONS

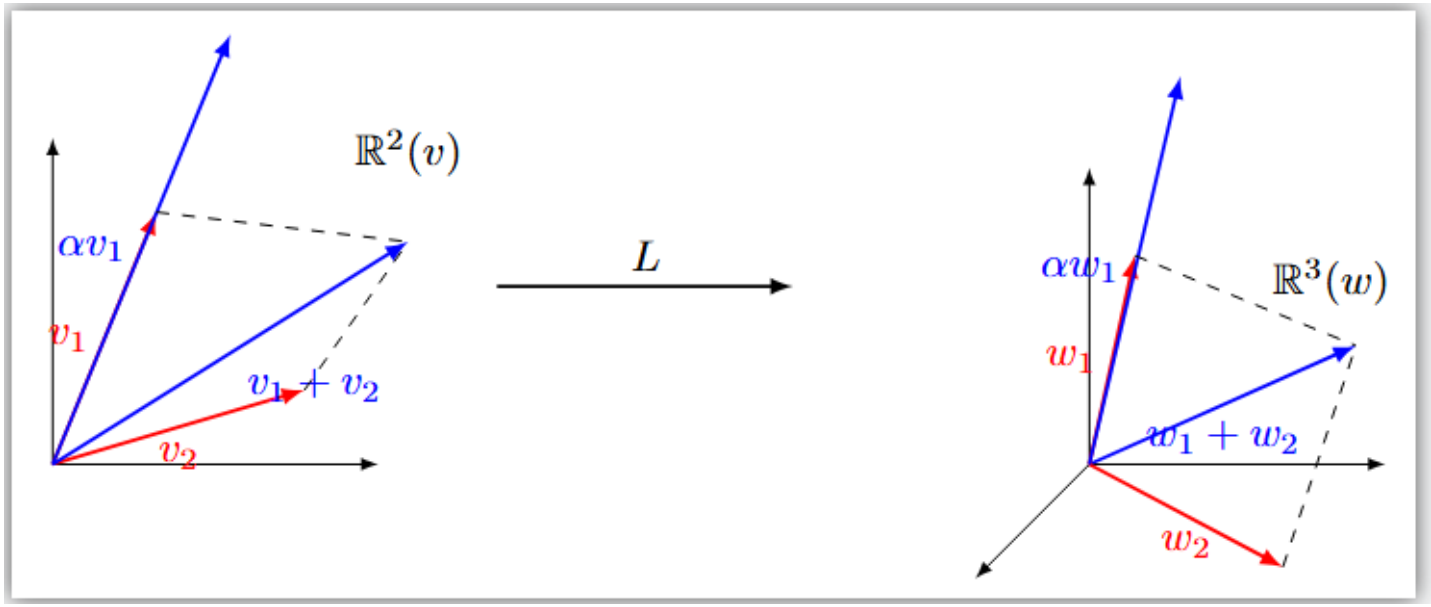
Definition:

A mapping L from a vector space V into a vector space W, denoted by

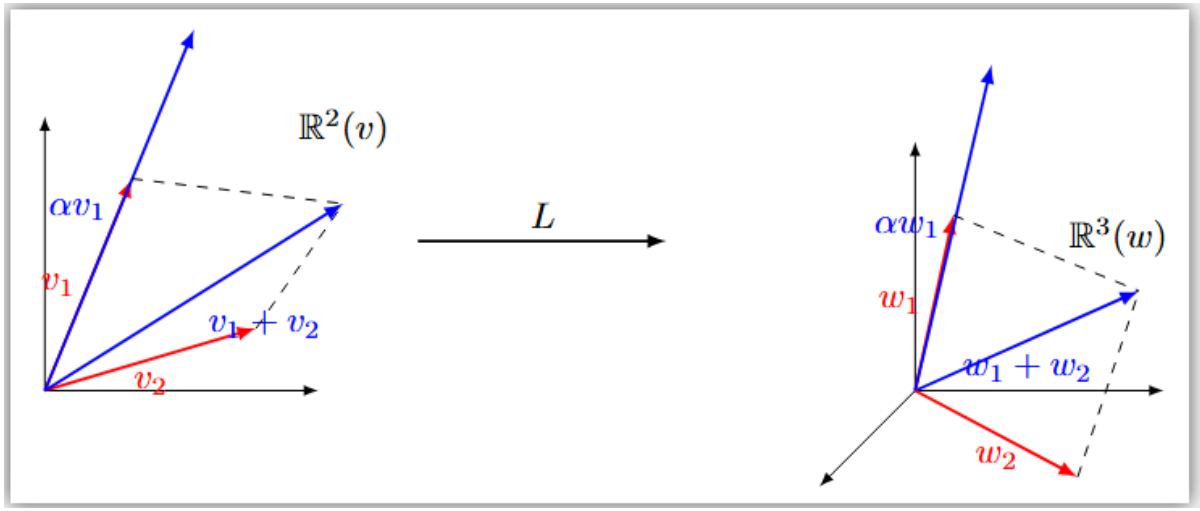
L: $V \rightarrow W$, is said to be a linear transformation if it satisfies two properties for all vectors v_1 and v_2 in the vector space V and all scalars $\propto$,

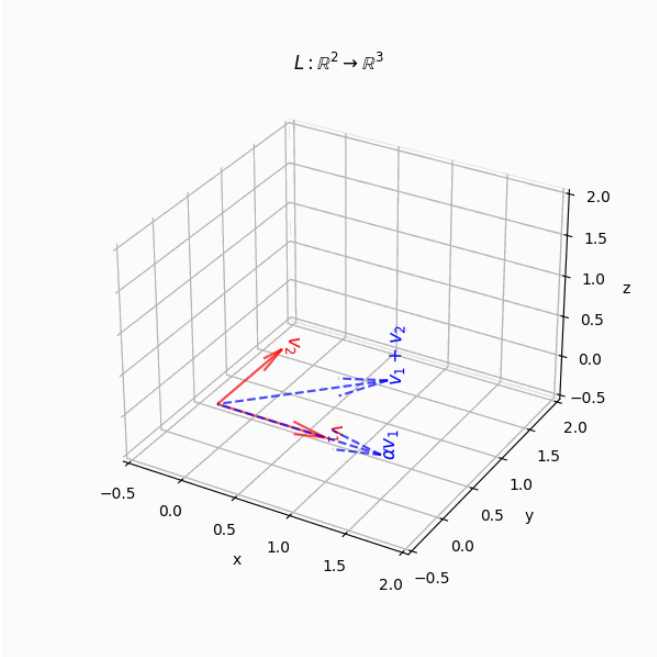
1. Additivity: L $(v_1 + v_2)$ = L (v_1) + L (v_2)

2. Scalar Multiplication: L$(\propto v_1)$ = $\propto$ L(v_1)



LINEAR TRANSFORMATIONS





Note:

1. L is a linear transformation if $L(\propto v_1 + \beta v_2) = \propto L(v_1) + \beta L(v_2)$, for any v_1 and v_2 in the vector space V and all scalars $\propto$ and β.

2. If $L: V \rightarrow V$ we will refer to a linear transformation L as linear operator on V

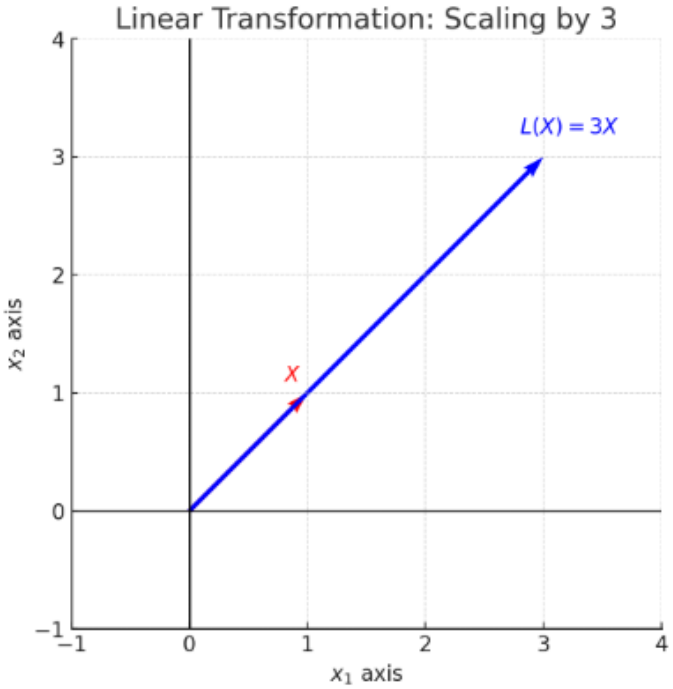
That is, a linear operator is a linear transformation from a vector space to itself.

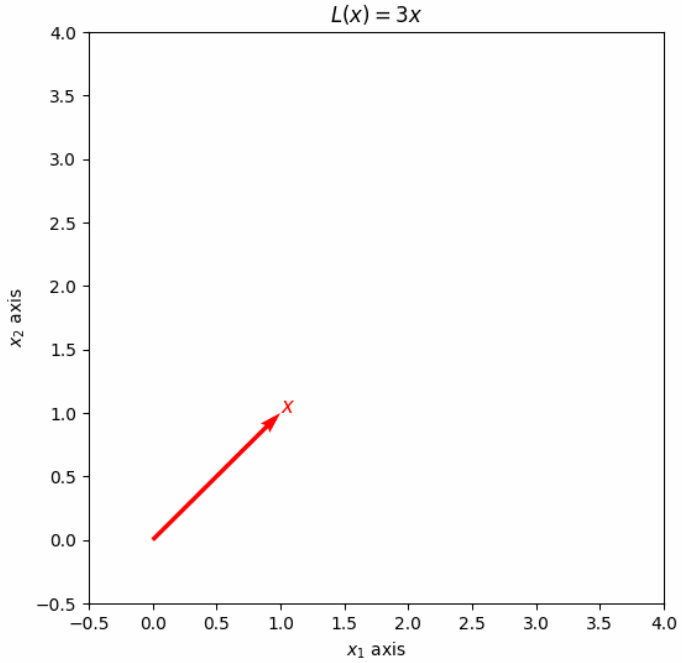
EXAMPLE OF A LINEAR TRANSFORMATION:

1. Let L be the operator defined by $L(X) = 3X$ for each x $\in \mathbb{R}^2$

>> It is a linear operator because

$$L(\propto X + \beta\ Y) = 3(\propto X + \beta\ Y) = \propto(3X) + \beta(3Y) = \propto L(X) + \beta\ L(Y)$$





In general, the linear operator $L(X) = \propto X$ can be thought of as a stretching or shrinking by a factor of $\propto$.

EXAMPLE OF A LINEAR TRANSFORMATION:

2. Consider the mapping L defined by $L(X) = x_1 e_1$ for each $X \in \mathbb{R}^2$. Thus ,

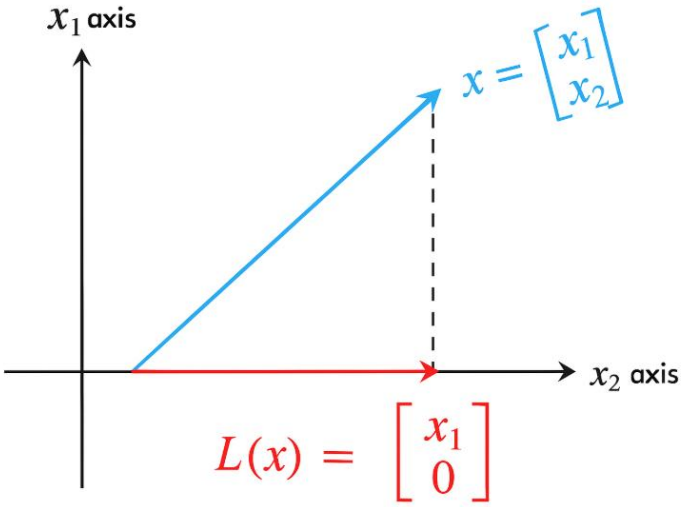
if $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $L(X) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.

>> It is a Linear operator because for all $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and Y= $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, for all scalar α and β.

$$L(\alpha X + \beta Y) = L\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ 0 \end{bmatrix} = (\alpha x_1 + \beta y_1)e_1$$

$$= \alpha(x_1 e_1) + \beta(y_1 e_1)$$

$$= \alpha L(X) + \beta L(Y).$$



We can think L as a projection onto the x_1 -axis

EXAMPLE OF A LINEAR TRANSFORMATION:

3. Suppose transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as follows: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -3y \end{bmatrix}$ verify that T is a linear transformation.

>> For any two vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ and scalars α, β in $\mathbb{R}^2$.

To show that T is a linear transformation we need to prove that

$$T\left(\alpha\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \beta\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

LHS,

$$T\left(\alpha\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \beta\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix} + \begin{bmatrix} \beta x_2 \\ \beta y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 & +\beta x_2 \\ \alpha y_2 & +\beta y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(\alpha x_1 + \beta x_2) \\ -3(\alpha y_1 + \beta y_2) \end{bmatrix}$$

RHS,

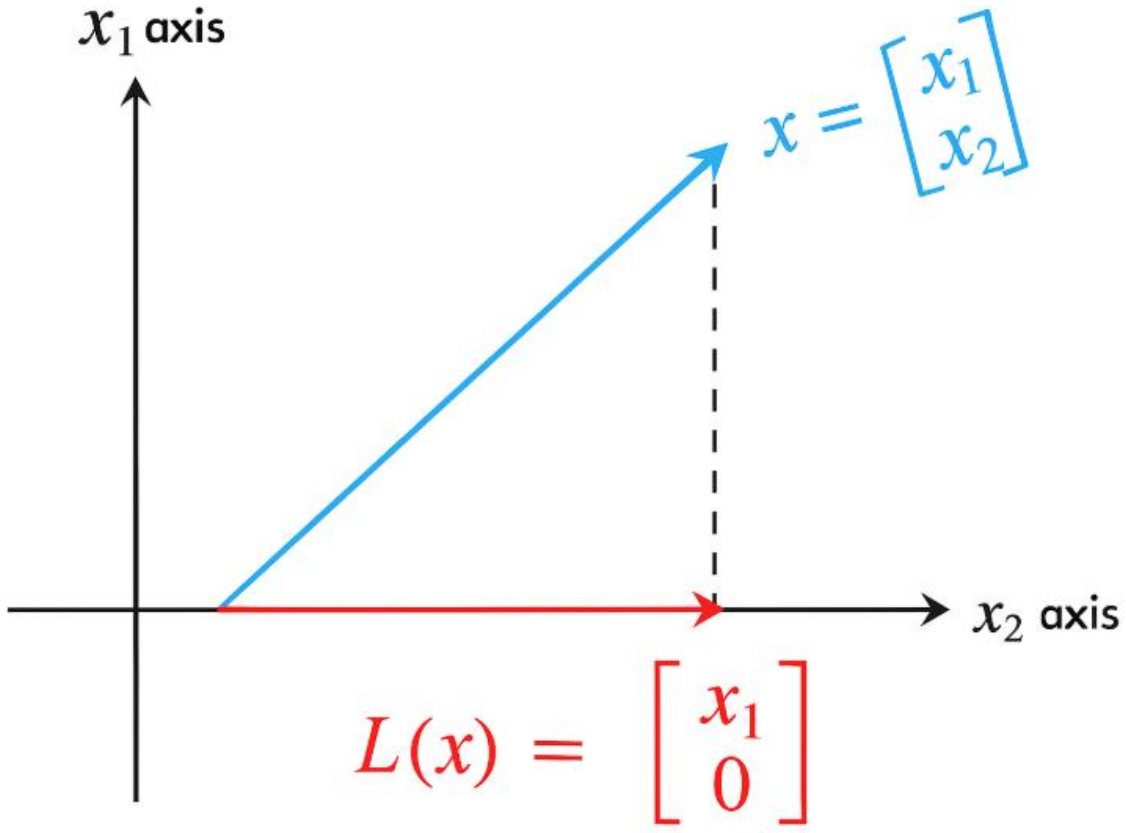
$$\alpha\, T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + \beta\, T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \alpha\begin{bmatrix} 2x_1 \\ -3y_1 \end{bmatrix} + \beta\begin{bmatrix} 2x_2 \\ -3y_2 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1 + \beta x_2) \\ -3(\alpha y_1 + \beta y_2) \end{bmatrix}$$

LHS = RHS. Hence T is a linear transformation.

CO1 Questions:

1. Interpret the transformation $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ geometrically.

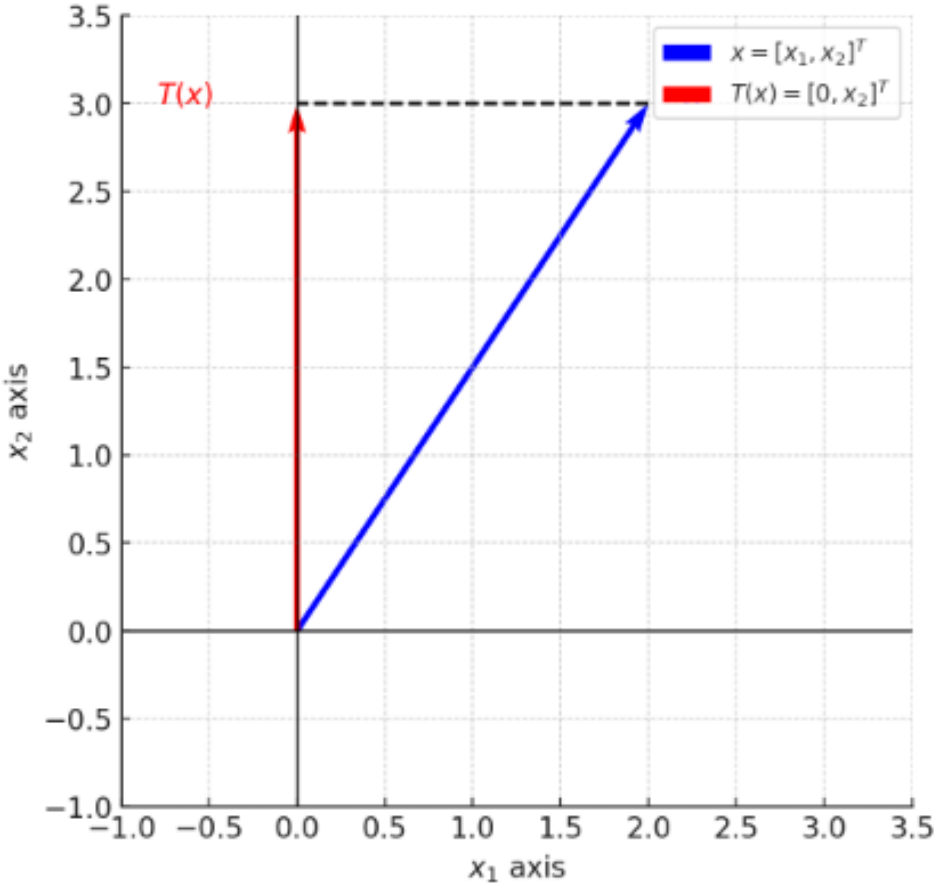
Ans:



CO1 Questions:

2. Interpret the transformation $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ geometrically.

Ans:



MITE
Invent Solutions

Reinforcement (Review) Questions:

Q1. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ x + 3y \end{bmatrix}$.

Is T a linear transformation?

A) Yes, because it scales and shifts vectors

B) Yes, because addition and scalar multiplication are preserved

C) No, because the constant $+3$ breaks linearity

D) No, because it maps from $\mathbb{R}^2$ to itself

Answer: C) No, because the constant $+3$ breaks linearity

Q2. A transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y \\ 2y \\ z + x \end{bmatrix}$.

Is it linear?

A) Yes

B) No, because of the $-y$ term

C) No, because of the $+x$ term

D) No, because z is not scaled

Answer: A) (all terms are linear combinations, no constants).

Reinforcement (Review) Questions:

Q3. Which of the following **must always be true** for a linear transformation T?

A) $T(0) = 0$

B) $T(x + y) = T(x) + T(y)$

C) $T(cx) = cT(x)$

D) All of the above

Answer: D) All of the above

Q4. Suppose $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. Then the image of any line through the origin is:

A) A parabola

B) A shifted line

C) A line through the origin (or a point at 0)

D) A circle

Answer: C) A line through the origin (or a point at 0)

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION

Diagnostic (Preview) Questions:

Q1. The order of a matrix with 3 rows and 5 columns is:

a) 5×3

b) 3×5

c) 15

d) None

Answer: b

Q2. Which of the following is NOT a type of matrix?

a) Row matrix

b) Column matrix

c) Identity matrix

d) Trigonometric matrix

Answer: d

Q3. If I is the identity matrix, then $AI = IA =$?

a) 0

b) A

c) I

d) Undefined

Answer: b

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION

Diagnostic (Preview) Questions:

Q4. A linear combination of vectors means:
a) Adding vectors only
b) Multiplying vectors
c) A sum of scalar multiples of vectors
d) None

Answer: c

Q5. The maximum number of linearly independent vectors in $\mathbb{R}^n$ is _____.

Answer: n

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION

Let L be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is matrix of order $m \times n$

such that $L(X) = AX$ where $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}$ for any $X \in \mathbb{R}$ and ,

$$L(e_i) = \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{mi} \end{bmatrix} , \text{i} = 1, 2, 3, \dots, n$$

Definition: If $X = x_1e_1 + x_2e_2 + \dots + x_ne_n$ is an arbitrary vector in $\mathbb{R}^n$

$$L(X) = x_1L(e_1) + x_2L(e_2) + \dots + x_nL(e_n) = [\, L(e_1), L(e_2), \dots, L(e_n) \,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ if}$$

$$L(e_i) = \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{mi} \end{bmatrix} \text{ then } L(X) = AX.$$

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION

Example:

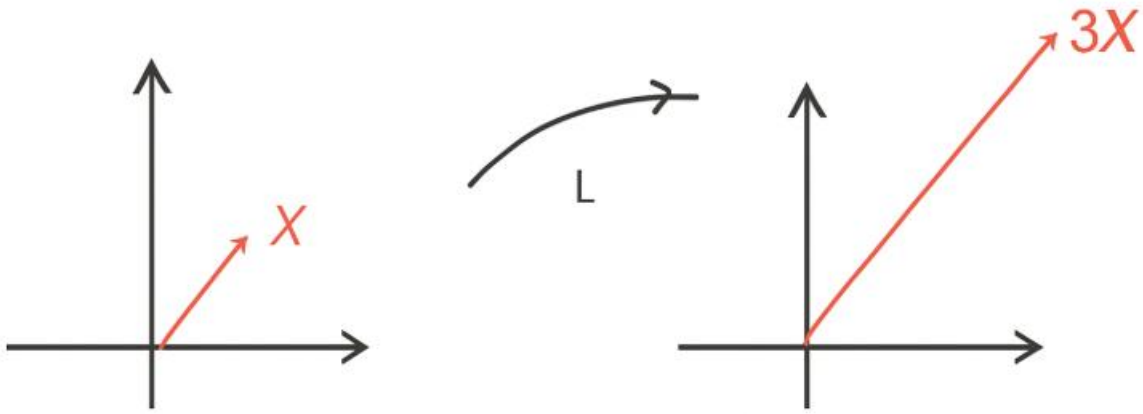
1. $L(X) = 3X$

Here $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, X= $x_1 e_1 + x_2 e_2$

$L(X) = L(x_1 e_1 + x_2 e_2) = x_1 L(e_1) + x_2 L(e_2)$

And $AX = [\ L(e_1)\ \ L(e_2)\ ] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Thus, matrix representation of the given linear transformation is A= $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



MITE
Invent Solutions

MATRIX REPRESENTATIONS OF A LINEAR TRANSFORMATION

2. Let $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $L(X) = x_1 e_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. To find matrix representation A.

$L(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $L(e_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- Interactive: A 2 × 3 matrix: https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?show=false
- Interactive: A 3 × 2 matrix: https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0:0,1:1,0&range2=5&closed=true&show=false
- Projection onto the x y-plane: https://textbooks.math.gatech.edu/ila/demos/Axequalsb.html?mat=1,0,0:0,1,0:0,0,0&range2=5&closed=true

PROBLEMS

1. Find the matrix of linear transformation $T: V_2(\mathbb{R}) \rightarrow V_3(\mathbb{R})$

defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \\ 3x-y \end{bmatrix}$ with respect to a standard basis.

Solution: Let $\{e_1, e_2\}$ and $\{f_1, f_2, f_3\}$ be the standard basis of $V_2(\mathbb{R})$ and $V_3(\mathbb{R})$ respectively.

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now,

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 1.f_1 + 1.f_2 + 3.f_3$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1.f_1 + 0.f_2 - 1.f_3$$

$$\therefore A_T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

PROBLEMS

2. If T is a mapping from $V_2(\mathbb{R})$ into $V_2(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta & -y\sin\theta \\ x\sin\theta & +y\cos\theta \end{bmatrix}$$

Show that T is a linear transformation.

Solution: Let $\propto = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two arbitrary elements of $V_2(\mathbb{R})$, Let $c \in \mathbb{R}$ be any scalar.

Consider $T(c\propto + d\beta) = T\left(\begin{bmatrix} cx_1 + dy_1 \\ cx_2 + dy_2 \end{bmatrix}\right)$

$$= \begin{bmatrix} (cx_1 + dy_1)\cos\theta - (cx_2 + dy_2)\sin\theta \\ (cx_1 + dy_1)\sin\theta + (cx_2 + dy_2)\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} c(x_1\cos\theta - x_2\sin\theta) + d(y_1\cos\theta - y_2\sin\theta) \\ c(x_1\sin\theta + x_2\cos\theta) + d(y_1\sin\theta + y_2\cos\theta) \end{bmatrix}$$

$$= c\begin{bmatrix} (x_1\cos\theta - x_2\sin\theta) \\ (x_1\sin\theta + x_2\cos\theta) \end{bmatrix} + d\begin{bmatrix} (y_1\cos\theta - y_2\sin\theta) \\ (y_1\sin\theta + y_2\cos\theta) \end{bmatrix}$$

$= c\,T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + dT\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = c\,T(\propto) + dT(\beta)$. Hence T is a linear transformation.

PRACTICE PROBLEMS

1. If $T: V_1(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ **is defined by** $T(x) = \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix}$

Verify whether T is linear or not.

2. If $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ **defined by** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + 3y \end{bmatrix}$.

Verify whether T is linear or not.

SOME INTERESTING TRANSFORMATION OF $x \in \mathbb{R}^2$

i) $I_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$ This is called identity transformation.

Example:

$$I_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = x$$

- https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1,0,0,1&closed=true&pic=theo11.jpg

ii)) $R_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = y$

$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is called a reflection.

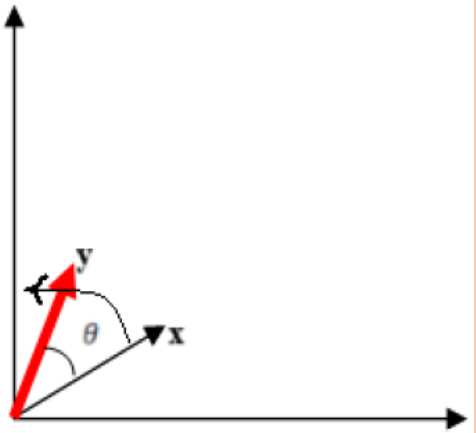
Example:

$$R_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = y$$

- https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=0,1,1,0&closed=true

iii) $R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

$$R_{\theta x} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 cos\theta - x_2 sin\theta \\ x_1 sin\theta + x_2 cos\theta \end{bmatrix} = y$$



R_θ Rotates the vector by an angle θ counter clockwise direction.

$$R_{(-\theta)} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

$R_{(-\theta)}$ is the inverse of R_θ

- ~~$\theta = 45°$~~: https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=0.707107,-0.707107,0.707107,0.707107&closed=true
- ~~$\theta = 90°$~~: https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=0,-1,1,0&closed=true&pic=theo8.jpg

iv) **Shear Transformation:**

a) Shear (Italic) along x axis: the transformation matrix is $\begin{bmatrix} 1 & k_x \\ 0 & 1 \end{bmatrix}$

- https://textbooks.math.gatech.edu/ila/demos/twobytwo.html?mat=1,1,0,1&closed=true&pic=sheep.jpg

b) Shear (Italic) along y axis: the transformation matrix is $\begin{bmatrix} 1 & 0 \\ k_y & 1 \end{bmatrix}$

PROBLEMS

1) Use the coordinates $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\6\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}$,and apply the transformation $A = \begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$ to each of the coordinates to find its transformation. Represent the transformation graphically and specify the nature of transformation.

Soln :

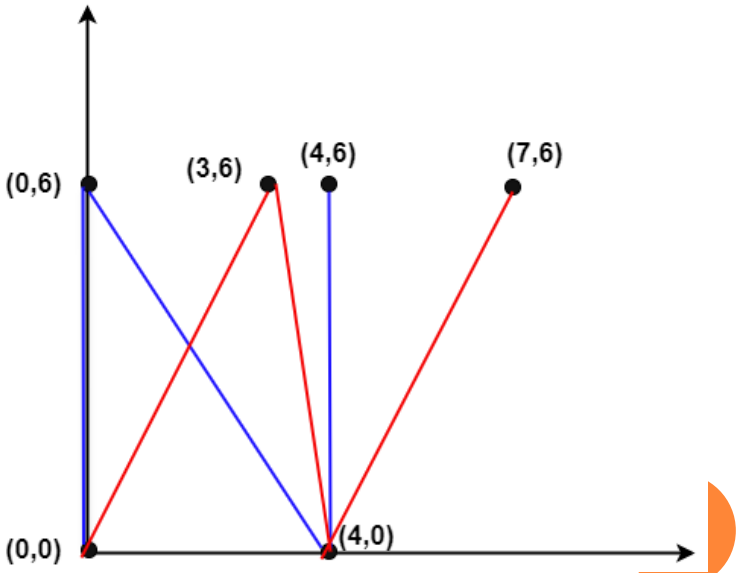
The Transformation Matrix is $A = \begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

The transformed coordinates are

$\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$; $\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}\begin{bmatrix}0\\6\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$;

$\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix}$; $\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}7\\6\end{bmatrix}$



PROBLEMS

v) Scaling Transformation:

2) Given a photograph with the coordinates $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}$ *and* $\begin{bmatrix}0\\0\end{bmatrix}$ of stamp size. Enlarge the photograph by 3 units in width and 5 units by height. Represent the transformation geometrically.

Soln:

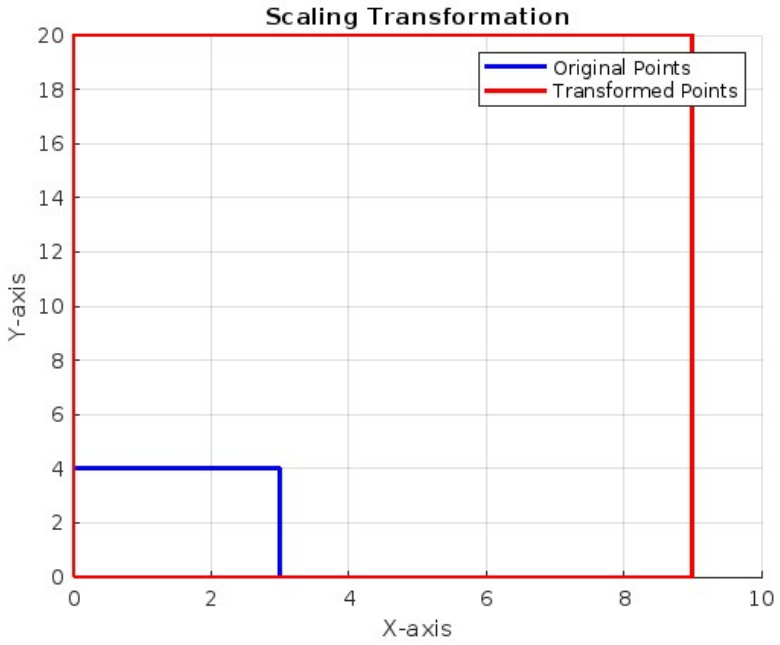
The Scaling transformation is T $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3x\\5y\end{bmatrix}$

The transformed coordinates are

$T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$; $T\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}9\\0\end{bmatrix}$;

$T\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}9\\20\end{bmatrix}$; $T\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}0\\20\end{bmatrix}$

$T\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$



-OR-

The Scaling transformation matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

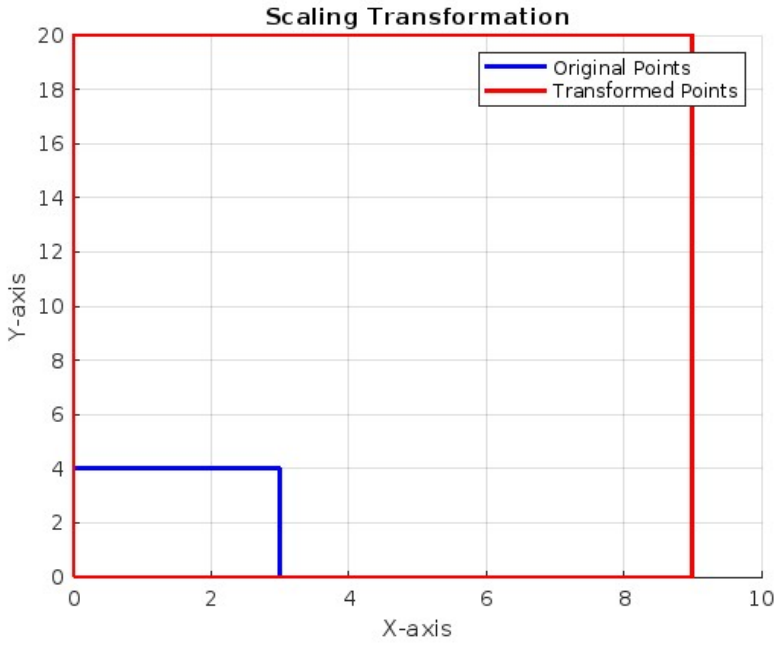
The transformed coordinates are

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 20 \end{bmatrix}; \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- https://textbooks.math.gatech.edu/ila/dempic=theo2.jpg



PROBLEMS

3) Given a photograph in the form of a square with the coordinates $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ *and* $\begin{bmatrix}0\\0\end{bmatrix}$ representing the vertices of photograph, apply a transformation to enlarge the photograph. The transformation is defined as $\boldsymbol{T}$: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$
Where $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{3}(\boldsymbol{x}, \boldsymbol{y})$. Specify the type of transformation and represent it graphically.

MITE
Invent Solutions

Reinforcement (Review) Questions:

1. If a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then T corresponds to:

a) Rotation
b) Reflection
c) Scaling
d) Shear

Answer: c) Scaling
Explanation: The diagonal entries stretch the x-axis by factor 2 and the y-axis by factor 3.

2. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents which transformation in $\mathbb{R}^2$?

a) Reflection about x-axis
b) 90° counter-clockwise rotation
c) 90° clockwise rotation
d) Projection onto y-axis

Answer: b) 90° counter-clockwise rotation
Explanation: This is the standard 2D rotation matrix for +90°.

Reinforcement (Review) Questions:

Q3. Which type of linear transformation is represented by the matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, k \neq 0$

a) Scaling
b) Shear
c) Projection
d) Rotation

Answer: b) Shear
Explanation: Non-zero off-diagonal entry with 1's on the diagonal produces a shear transformation.

Q4. If the determinant of a transformation matrix is **0**, the transformation is:
a) Always invertible
b) Not invertible (singular)
c) A rotation
d) A scaling
Answer: b) Not invertible (singular)
Explanation: means the transformation compresses space into a lower dimension (e.g., projection), so it cannot be inverted.

MITE
Invent Solutions

Reinforcement (Review) Questions:

Q5. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ represents which transformation?

a) Reflection about x-axis
b) Projection onto x-axis
c) Scaling by factor 0
d) Shear

Answer: b) Projection onto x-axis
Explanation: The y-coordinate is mapped to 0, so all points are projected onto the x-axis.

Q6. If a linear transformation preserves both **lengths** and **angles**, it must be:
a) Shear
b) Rotation or reflection
c) Scaling
d) Projection

Answer: b) Rotation or reflection
Explanation: Only orthogonal transformations preserve both lengths and angles.

MITE
Invent Solutions

Reinforcement (Review) Questions:

Q7. The eigenvalues of a pure rotation matrix in $\mathbb{R}^2$ are:
a) Always real
b) Always 0
c) Complex conjugates on the unit circle
d) Equal to the rotation angle

Answer: c) Complex conjugates on the unit circle
Explanation: A 2D rotation matrix has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$.

MITE
Invent Solutions

CHANGE OF BASIS

Let us consider a vector space V with two different bases, $B = \{ v_1 , v_2 ,\ldots, v_n\}$ and $B' = \{u_1, u_2 ,\ldots, u_n\}$.

Let v be expressed as: $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

Then, one can find the representation of the same vector in terms of the basis B' by solving a system of linear equations.

Definition: The standard basis for $\mathbb{R}^2$ is $\{e_1, e_2\}$. So, any vector in $\mathbb{R}^2$ can be expressed as a linear combination $X = x_1e_1 + x_2e_2$. The scalars x_1 and x_2 are called the 'co-ordinates of X' w.r.t. $\{e_1, e_2\}$. And $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is called the 'co-ordinate vector of X'.

Let {y,z} be any other basis in $\mathbb{R}^2$. Then X can be expressed uniquely as: X = ay+bz.

Here, the scalars a and b are the 'coordinates of X' w.r.t. {y,z} and $\begin{bmatrix} a \\ b \end{bmatrix}$ is called the 'co-ordinate vector of X'.

Note: The co-ordinate vector of X w.r.t. the basis {z,y} is $\begin{bmatrix} b \\ a \end{bmatrix}$.

To avoid confusion, we consider 'ordered basis'. In the ordered basis {y,z}, y is the first basis vector and z is the second basis vector.

MITE

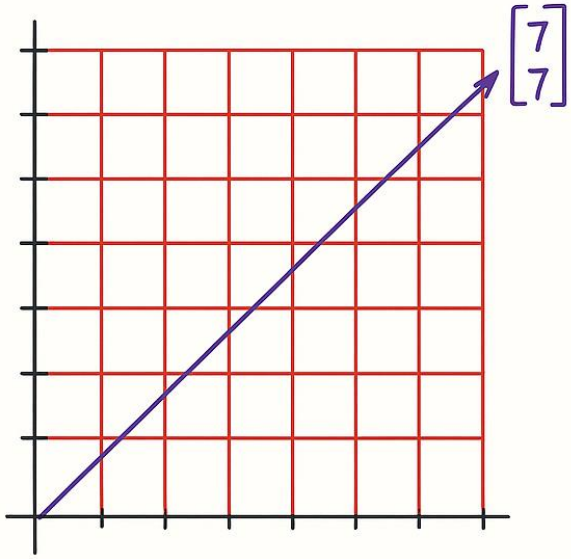
Invent Solutions

Example:

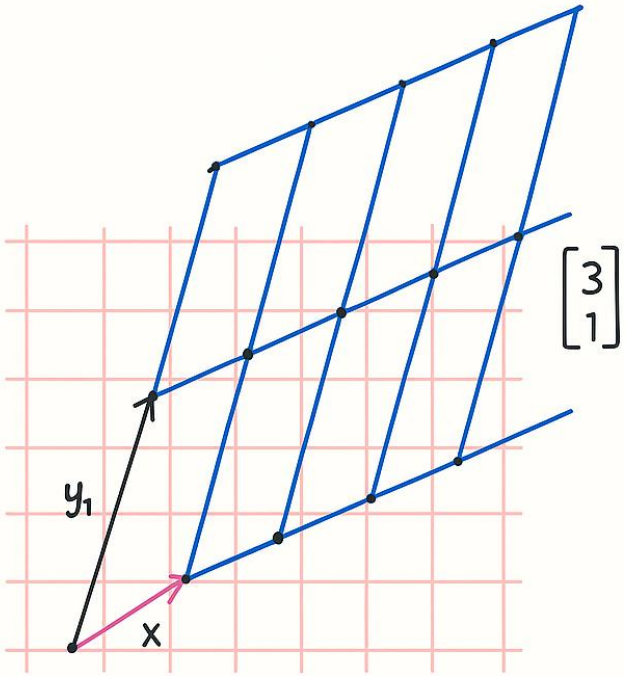
Let $\{y_1, y_2\}$ be the basis in $\mathbb{R}^2$, where $\boldsymbol{y_1} = \begin{bmatrix}\mathbf{2}\\\mathbf{1}\end{bmatrix}$ ***and*** $\boldsymbol{y_2} = \begin{bmatrix}\mathbf{1}\\\mathbf{4}\end{bmatrix}$. Let $\boldsymbol{X} = \begin{bmatrix}\mathbf{7}\\\mathbf{7}\end{bmatrix}$ be a vector in $\mathbb{R}^2$.

Then, the coordinate vector of X w.r.t. $\{y_1, y_2\}$ is $\begin{bmatrix}\mathbf{3}\\\mathbf{1}\end{bmatrix}$ (since $X = 3\ y_1 + 1\ y_2$).

Geometric interpretation



$\begin{bmatrix} 7 \\ 7 \end{bmatrix}$ coordinate vector wrt std basis e_1, e_2



$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is coordinate vector of x wrt y_1, y_2

MITE

Invent Solutions

Changing co-ordinates:

Problems:

1. Consider the following basis of $\mathbb{R}^2$, $E = \{e_1, e_2\} = \{[1\ 0]^T, [0\ 1]^T\}$ and $S = \{u_1, u_2\} = \{[1\ 3]^T, [1\ 4]^T\}$.

i) Find the change of basis matrix P from the matrix E to S

ii) Find the change of basis matrix Q from the matrix S to E

iii) Find the coordinate of $\boldsymbol{v} = \begin{bmatrix} \mathbf{5} \\ \mathbf{-3} \end{bmatrix}$ relative to S

SOLUTION:

i) To find the change of basis matrix from basis E to basis S, you need to express each vector in basis E as a linear combination of vectors in basis S and then arrange the coefficients in a matrix. Let's denote the change of basis matrix from E to S as P.

Given basis vectors: $E = \{[1\ 0]^T, [0\ 1]^T\}$, $S = \{[1\ 3]^T, [1\ 4]^T\}$.

Let $P = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix}$.

For $[1\ 0]^T$ in basis E, express it as a linear combination of basis vectors in S:

$$[1\ 0]^T = p_{11}[1\ 3]^T + p_{12}[1\ 4]^T.$$

This leads to the system of equations:

$1 = p_{11} + p_{12}$ and $0 = 3p_{11} + 4p_{12}$

Solving this system will give you the values of $p_{11} = 4, p_{12} = -3$

Repeat the process for

$$[0\ 1]^T = p_{21}[1\ 3]^T + p_{22}[1\ 4]^T$$

This leads to the system of equations:

$0 = p_{21} + p_{22}$ and $1 = 3p_{21} + 4p_{22}$

Solving this system will give you the values of $p_{21} = -1, p_{22} = 1$

The resulting matrix P will be the change of basis matrix from E to S.

$$P = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

ii) Now, to find the change of basis matrix from S to E, you can follow a similar procedure.

Let $q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix}$.

For $[1\ 3]^T$ in basis S, express it as a linear combination of basis vectors in E:

$[1\ 3]^T = q_{11}[1\ 0]^T + q_{12}[0\ 1]^T$.

This leads to the system of equations: $1 = q_{11}$ and $3 = q_{12}$ which gives the values of $q_{11}$and q_{12}.

Repeat the process for$[1\ 4]^T = q_{21}[1\ 0]^T + q_{22}[0\ 1]^T$.

This leads to the system of equations: $1 = q_{21}$and $4 = q_{22}$ gives the values of $q_{21}$and q_{22}

The resulting matrix Q will be the change of basis matrix from S to E.

$$Q = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

iii) To find the coordinates of the vector $v = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ with respect to the basis $S = \{u_1, u_2\} = \{[1\ 3]^T, [1\ 4]^T\}$, we need to express v as a linear combination of $u_1\ and\ u_2$. The coordinates $[a, b]$ of v with respect to the basis S are the coefficients in this linear combination. The given basis vectors are: $u_1 = [1\ 3]^T$ and $u_2 = [1\ 4]^T$, we want to find a and b such that: $v = au_1 + bu_2$.

Substitute the components of v and the basis vectors: $\begin{bmatrix} 5 \\ -3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Now, set up a system of equations and solve for a and b:

$a + b = 5;\ 3a + 4b = -3.$

On solving, $a = 23, b = -18.$

MITE
Invent Solutions

HOMEWORK PROBLEMS

1. The vector $u_1 = [1\ 2\ 0]^T$, $u_2 = [1\ 3\ 2]^T$, $u_3 = [0\ 1\ 3]^T$ form a basis S of $\mathbb{R}^3$.

i) Find the change of basis matrix P from the usual basis E= $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ to the basis S.

ii) Find the change of basis matrix Q from the basis S to the usual basis E.

Solution: P = $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$

PRACTICE PROBLEMS

1. Let $V = \mathbb{R}^3$ and let $S = \{v_1, v_2, v_3\}$ and $T = \{w_1, w_2, w_3\}$ be the basis of $\mathbb{R}^3$, where $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, v_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}, w_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, w_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}, w_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

   Obtain the transition matrix from T to S

   Solution: $\begin{bmatrix}\frac{3}{2} & 0 & 1\\ -2 & 1 & -1\\ -\frac{9}{2} & 2 & -2\end{bmatrix}$

2. Let $V = \mathbb{R}^3$ and let $E = \{v_1, v_2, v_3\}$ and $F = \{u_1, u_2, u_3\}$ be the basis of $\mathbb{R}^3$, where $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, v_2 = \begin{bmatrix}2\\3\\2\end{bmatrix}, v_3 = \begin{bmatrix}1\\5\\4\end{bmatrix}, u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, u_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}, u_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$

   Let $X = 3v_1 + 2v_2 - v_3; Y = v_1 - 3v_2 + 2v_3$. Obtain the transition matrix from E to F, use it to find coordinate vectors of X and Y with respect to the ordered basis F

   Solution: Transition matrix = $\begin{bmatrix}1 & 1 & -3\\ -1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}$, $[X]_F = \begin{bmatrix}8\\-5\\3\end{bmatrix}$, $[Y]_F = \begin{bmatrix}-8\\2\\3\end{bmatrix}$

1. Find the transition matrix corresponding to the change of basis from $\{v_1, v_2\}$ to $\{u_1, u_2\}$, where $v_1 = \begin{bmatrix}5\\2\end{bmatrix}, v_2 = \begin{bmatrix}7\\3\end{bmatrix}, u_1 = \begin{bmatrix}3\\2\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\end{bmatrix}$

   Solution: $\begin{bmatrix}3 & 4\\-4 & -5\end{bmatrix}$

2. Let $u_1 = \begin{bmatrix}1\\-1\end{bmatrix}, u_2 = \begin{bmatrix}-2\\3\end{bmatrix}$. Find the transition matrix from $\{e_1, e_2\}$ to $\{u_1, u_2\}$ and determine the coordinate of $X = \begin{bmatrix}1\\2\end{bmatrix}$ with respect to $\{u_1, u_2\}$

   Solution: Transition matrix = $\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$, $[X]_{\{u_1,u_2\}} = \begin{bmatrix}7\\3\end{bmatrix}$

3. Let $u_1 = \begin{bmatrix}3\\2\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\end{bmatrix}, X = \begin{bmatrix}7\\4\end{bmatrix}$. Find the coordinate of X with respect to $\{u_1, u_2\}$

   Solution: $[X]_{\{u_1,u_2\}} = \begin{bmatrix}3\\-2\end{bmatrix}$

MITE
Invent Solutions

Reinforcement (Review) Questions:

Q1. Which of the following is **always true** for a change-of-basis matrix P?

a) P is orthogonal
b) P is invertible
c) P is symmetric
d) P is diagonal

Answer: b) P**is invertible**
Explanation: Since bases are linearly independent, Pmust be invertible.

Q2. If $T: V \to V$is a linear transformation, and $A$and $B$are its matrix representations in different bases, then they are related by:

a) $B = A$
b) $B = P^{-1}AP$
c) $B = PAP^{-1}$
d) None

Answer: b) $B = P^{-1}AP$
Explanation: This is the similarity transformation formula when changing basis.

MITE

Invent Solutions

Reinforcement (Review) Questions:

1.The matrix whose columns are the coordinates of new basis vectors in the old basis is called the _____.

1.Answer: Change-of-basis matrix

2.A change-of-basis matrix is always _____ because it maps one basis to another.

2.Answer: Invertible

3.If $P$is the change-of-basis matrix from basis $B$to basis C, then the inverse P^{-1}converts coordinates from _____ to _____.

3.Answer: From basis $C$to basis B

4.If a linear transformation has matrix Ain basis B, then in another basis $C$the matrix becomes _____.

4.Answer: $P^{-1}AP$

5.A vector expressed in a non-standard basis is written as _____ coordinates.

5.Answer: Coordinate vector (relative to that basis)

KERNEL AND RANGE

Theorem: Let T: $U \rightarrow V$ be a linear transformation. Let 0_u and 0_v be the zero vectors of U and V. Then,

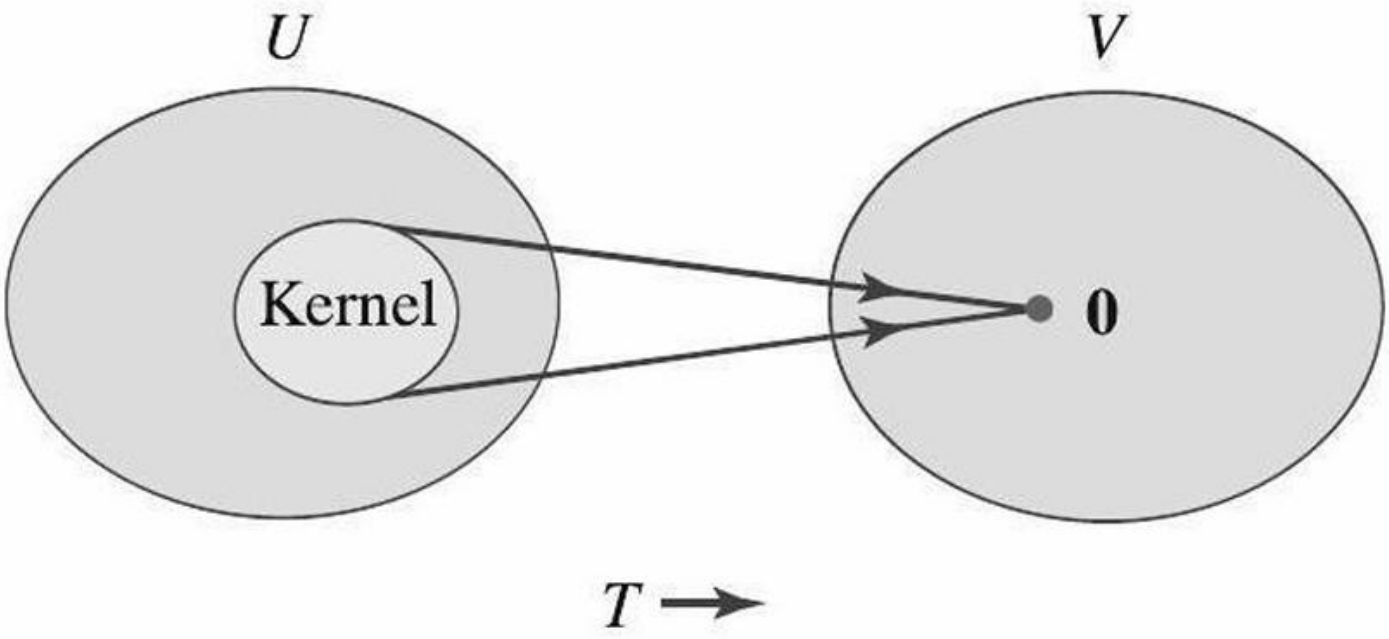
$$T(0_u) = 0_v \, .$$

That is, a linear transformation maps a zero vector into a zero vector.

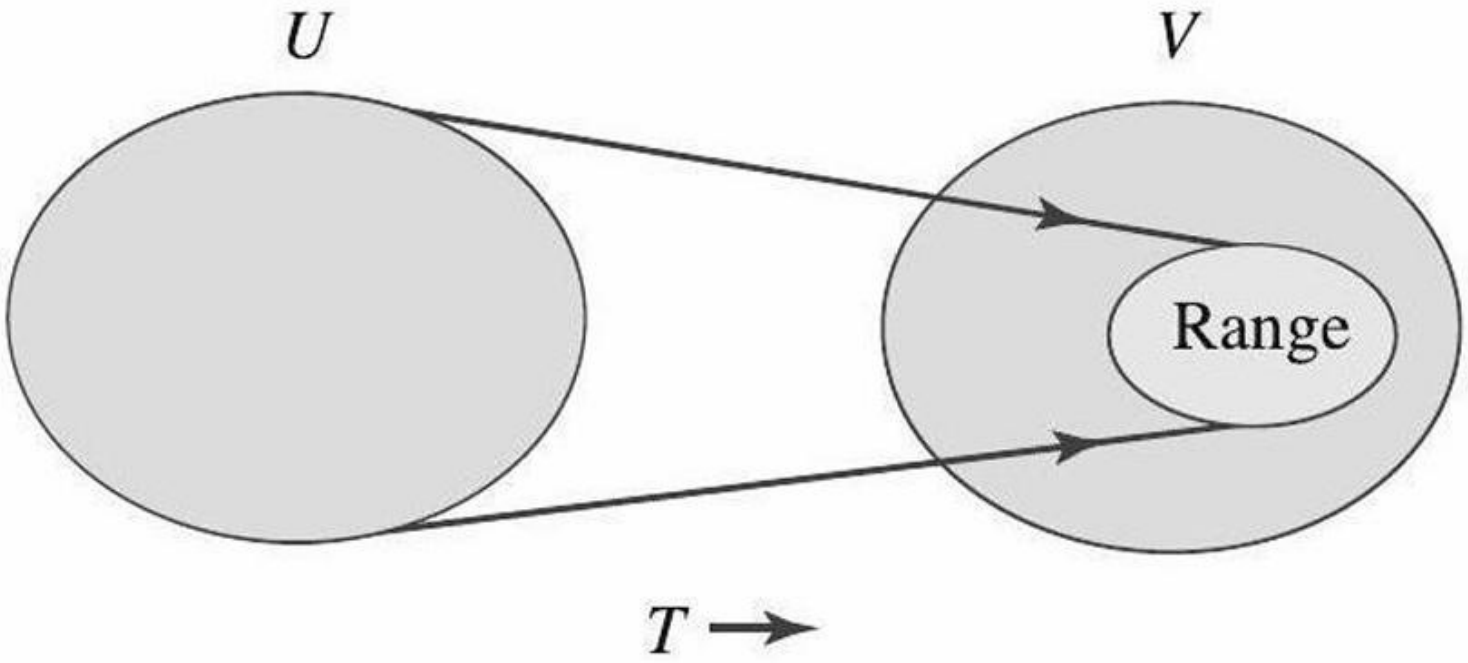
Definition: Let $T : U \rightarrow V$ be a linear transformation.

The set of vectors in U that are mapped into the zero vector of V is called the **kernel of T**. The kernel is denoted by $ker\ (T)$. The kernel is often called the null space and denoted null (T).

The set of vectors in V that are the images of vectors in U is called the **range of T**. The range is denoted by range(T).



Kernel: All vectors in U that are mapped into 0.



Range: All vectors in V that are images of vectors in U.

Theorem: Let $T:U \rightarrow V$ be a linear transformation.

i) The kernel of T is a subspace of U.

ii) The range of T is a subspace of V.

MITE

Invent Solutions

Example: Find the kernel and range of the linear operator,

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Solution: Since the linear operator T maps $\mathbb{R}^3$ into $\mathbb{R}^3$, the kernel and range will both be subspaces of $\mathbb{R}^3$.

Kernel: $\ker(T)$ is the subset that is mapped into $(0,0,0)$. We see that

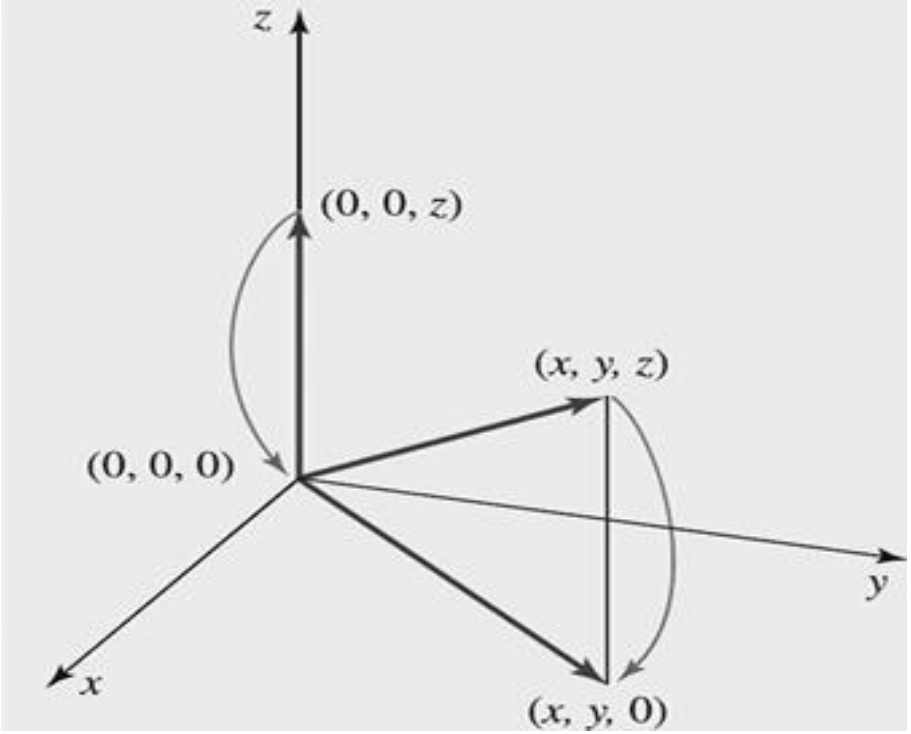
$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ if } x = 0, y = 0.$$

Thus $\ker(T)$ is the set of all vectors of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$. We express this as

$$\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R} \right\}$$

Geometrically, $\ker(T)$ is the set of all vectors that lie on the z-axis.

We illustrate the range of the transformation in Figure. Observe that T projects the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ into the vector $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in the xy-plane. T projects all vectors onto the xy-plane. T is an example of a projection operator.



MITE

Invent Solutions

Applications:

The world in which we live has three spatial dimensions.

When we observe an object, however, we get a two-dimensional impression of that object, the view changing from location to location. Projections can be used to illustrate what three-dimensional objects look like from various locations.

They are used in the architecture, the auto industry, and the aerospace industry.

MITE

Invent Solutions

Theorem: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be defined by $T(u) = Au$. The range of T is spanned by the column vectors of A.

Example: Determine the kernel and the range of the transformation defined by the following matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

MITE

Invent Solutions

SOLUTION:

A is a 3×3 matrix. Thus A defines a linear operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$T(\mathbf{x}) = A\mathbf{x}.$$

The elements of $\mathbb{R}^3$ are written in column matrix form for the purpose of matrix multiplication.

Kernel: The kernel will consist of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that

$$T(\mathbf{x}) = \mathbf{0}$$

Thus,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This matrix equation corresponds to the following system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$x_1 + x_2 + 4x_3 = 0$$

On solving this system, we get many solutions, $x_1 = -5r, x_2 = r, x_3 = r$. The kernel is thus the set of vectors of the form $\begin{bmatrix} -5r \\ \mathrm{r} \\ r \end{bmatrix}$.

$$\mathrm{Ker}(T) = \left\{ \begin{bmatrix} -5r \\ \mathrm{r} \\ r \end{bmatrix} | r \; is \; any \; real \; number \right\}$$

$\mathrm{Ker}(T)$ is a one-dimensional subspace of $\mathbb{R}^3$ with basis $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$.

MITE

Invent Solutions

Range: The range is spanned by the column vectors of A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ span the range of T.

Thus, the range of T is

$$\text{Range}(T) = span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$$

The vectors $(1,0,1)$ and $(2,-1,1)$ are also linearly independent. Range (T) is a two dimensional subspace of $\mathbb{R}^3$ with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$.

MITE

Invent Solutions

Rank-Nullity Theorem:

Let $T: U \rightarrow V$ be a linear transformation. Then,

$$\dim \ker(T) + \dim \text{range}(T) = \dim \text{domain}(T)$$

i.e., nullity T + rank T = dim U

MITE

Invent Solutions

Example:

1. Consider the following tranformation matrix A and the linear transformation T defined by A.

Find rank(A) , dim domain(T) , dim range(T) , dim ker(T), and nullity(T).

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Observe that the matrix A is in echelon form. The nonzero row vectors are linearly independent. Thus, $\text{rank}(A) = 2$. Since A is a 3×3 matrix, $T: \mathbb{R}^3 \to \mathbb{R}^3$. Thus $\dim \text{domain}(T) = 3$.

$$\dim \text{range(T)} = \text{rank(A)} = 2.$$

$$\dim \ker(\text{T}) = \dim \text{domain(T)} - \dim \text{range(T)} = 3 - 2 = 1$$

$$\therefore \text{nullity(T)} = \dim \text{kernel(T)} = 1.$$

MITE

Invent Solutions

2. Consider the transformation matrix

$$A = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$$

Here, the rank is 1, since the basis is{(3,-6)} (Range= span$\{(3,-6)\}$). The kernel of A is vectors such that $Av = 0$, which is $\{(-k/3, k)|$k is any real number$\}$ and has dimension 1. Hence the rank and nullity are both 1, and sum to 2, the number of columns in A.

MITE

Invent Solutions

3.This can be applied to non-square matrices as well. For instance, in the transformation matrix

$$A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 4 & 2 \end{bmatrix},$$

the rank is 2, spanned by the first two columns of A, and the kernel is a vector space spanned by {(22k/3, −7k/3, k)| k is any real number} that thus has dimension 1. As expected, the sum of the rank and nullity is thus 3, the number of columns in A.

MITE
Invent Solutions

4. $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$ defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 2x+z \end{bmatrix}$, verify the rank nullity theorem.

Solution: $T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Consider,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore r(T) = 3 \; and \; n(T) = 0$$

$$r(T) + n(T) = 3 + 0 = 3 = d[V_3(\mathbb{R})]$$

Theorem is verified.

MITE
Invent Solutions

PRACTICE PROBLEMS

1. $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ defined by $T(x, y, z) = (x + 2y + z, z - x, y + z)$, verify the rank nullity theorem.

Solution: $r(T) + n(T) = 2 + 1 = 3 = d[V_3(\mathbb{R})]$

2. $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix}$, find a basis for nullspace(A) and verify rank nullity theorem.

Solution: rank(A) + nullity(A) = 2 + 2 = 4 = n.

3. Determine the null space and verify the Rank-Nullity Theorem.

a. $A = [1 \quad 0 \quad -6 \quad -1]$ b. $B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ c. $C = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 4 \\ 1 & 1 & 0 \end{bmatrix}$

d. $D = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 2 & 9 & -1 & 7 \\ 2 & 8 & -2 & 6 \end{bmatrix}$.

MITE
Invent Solutions

Reinforcement (Review) Questions:

Q1. The **kernel** of a linear transformation $T: V \to W$ is:

a) The set of all outputs of T b) The set of all vectors in V mapped to 0 in W

c) The dimension of the image d) Always equal to V

Answer: b) The set of all vectors in V mapped to 0

Q2. The **range** (or image) of $T: V \to W$ is:

a) The set of all outputs $T(v)$ where $v \in V$ b) Always equal to V

c) The set of vectors mapped to zero d) The null space of T

Answer: a) The set of all outputs $T(v)$ where $v \in V$

Q3. For the linear transformation $T(x, y) = (x, 0)$ from $\mathbb{R}^2 \to \mathbb{R}^2$:

a) Kernel = $\{(0, y)\}$,Range = x-axis

b) Kernel = x-axis, Range = y-axis

c) Kernel = $\{(x, 0)\}$,Range = $\{(0, y)\}$

d) Both kernel and range are whole $\mathbb{R}^2$

Answer: a) Kernel = $\{(0, y)\}$,Range = x-axis

Q4. If the kernel of a transformation T is only the zero vector, then T is:

a) Not linear b) One-to-one (injective)

c) Onto (surjective) d) Zero transformation

Answer: b) One-to-one (injective)

MITE

Invent Solutions

Reinforcement (Review) Questions:

1.The kernel of a linear transformation is also called the _____.

1.Answer: Null space

2.The range of a transformation is also called the _____.

2.Answer: Image

3.If a transformation has a non-trivial kernel (more than just {0} ,(then the transformation is _____.

3.Answer: Not one-to-one (not injective)

4.The rank of a linear transformation is the **dimension of its** _____.

4.Answer: Range (image)

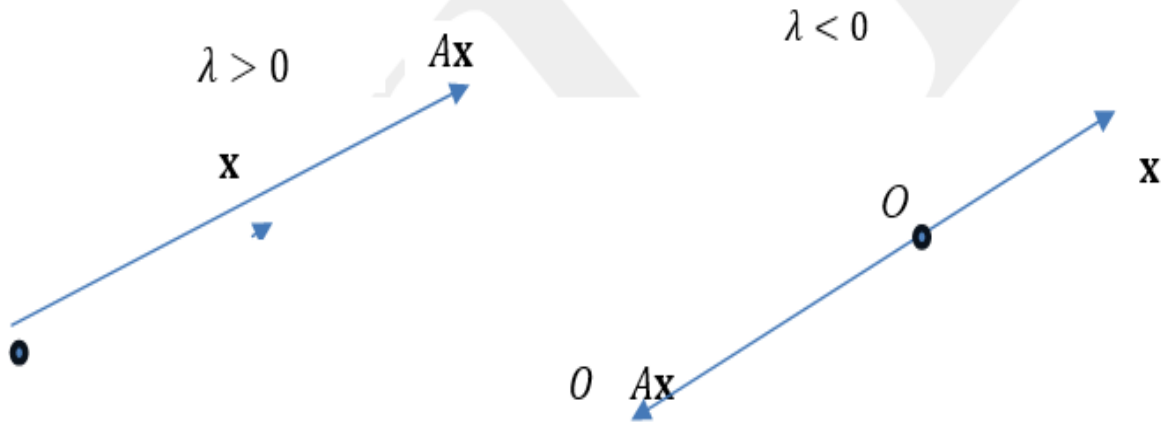
5.The nullity of a linear transformation is the **dimension of its** _____.

5.Answer: Kernel (null space)

EIGEN VALUES AND EIGEN VECTORS

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue of *A*** if there exist a non-zero vector x in $\mathbb{R}^n$ such that $Ax = \lambda x$. The vector x is called **eigenvector** corresponding to λ.

Let us look at the geometrical significance of an eigenvector that corresponds to a non-zero eigenvalue. The vector Ax is in the same or opposite direction as x, depending on the sign of λ. An eigenvector of A is thus a vector whose direction is unchanged or reversed when multiplied by A.



x is an eigenvector of A. A**x** is in the same or opposite direction as **x.**

RECALL: In Module 1, it was discussed how to calculate eigenvalues and eigenvectors using matrix determinants and matrix multiplication ($A\mathbf{v} = \lambda\mathbf{v}$).

Now in Module 3, one can observe that: Matrix multiplication is actually a geometric transformation of space. Eigenvectors uncover the natural coordinate axes where the transformation simplifies to pure stretching or shrinking, making complex linear mappings intuitive and easy to analyze.

MITE
Invent Solutions

Applications

1. Computer Science and Computer Graphics:

Computer Graphics: Matrices are used to represent transformations such as rotation, translation and scaling. Linear algebra is crucial for 3D graphics and animation.

Data Analysis: Techniques like Principal Component Analysis (PCA) use linear algebra for dimensionality reduction and feature extraction.

2. Engineering:

Linear transformations are used in circuit analysis and signal processing. For example, Fourier transforms and Laplace transforms are linear transformations that are essential in analyzing and designing electrical circuits.

3. Machine Learning and Data Science:

Regression Analysis: Linear regression is a common statistical method for predicting outcomes based on linear relationships between variables.

Neural Networks: The weights and biases in neural networks are often optimized using linear algebra

operations.

4. Economics and Finance:

Input-Output Models: Linear algebra is used to model and solve input-output systems, describing economic relationships.

Portfolio Optimization: Linear algebra is applied in optimizing investment portfolios.

5. Cryptography:

Public Key Cryptography: Algorithms like RSA use linear algebraic structures, such as modular arithmetic with matrices, for secure communication.

6. Biomedical Sciences:

Image Processing: Linear algebra is used in medical imaging for tasks like image reconstruction and feature extraction.

Population Modeling: Linear algebraic models are employed to analyze population dynamics in epidemiology.

7. Operations Research:

Linear Programming: Optimization problems, common in operations research, are often solved usinglinear algebra techniques.

MITE

Invent Solutions

THANK YOU