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NAAC A+ GRADE CGPA: 3.44

MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING

(A Unit of Rajalaxmi Education Trust ®, Mangalore)

Autonomous Institute affiliated to VTU, Belgavi, Approved by AICTE, New Delhi

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MODULE 4 : ORDINARY DIFFERENTIAL EQUATIONS

- **Differential Equations of first order:** Definitions, Geometrical meaning of a Differential Equation of the first order and first degree, Formation of a Differential Equation.

  Solution of Differential Equations of the first order and first degree: Variable Separable form, Linear Differential Equations, Applications to solve simple Engineering and Scientific problems.

- **Numerical solution of Ordinary Differential Equations:** Introduction, Modified Euler's method and Runge - Kutta method of fourth order to solve simple Engineering and Scientific problems.

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Learning Objective

- Establish a foundation of modeling dynamical systems to obtain their solution by Analytical and Numerical methods.

Imagine you have Rs.1,50,000 and want to invest it at an annual interest rate of 10.5%, compounded continuously. You want to know how much money you will have after a certain number of years(ex. 25 years). How would you calculate this?

OR

When you approach the bank they give the required data for any number of years/ for any amount deposited instantaneously how?

Manually?

How do they execute this?

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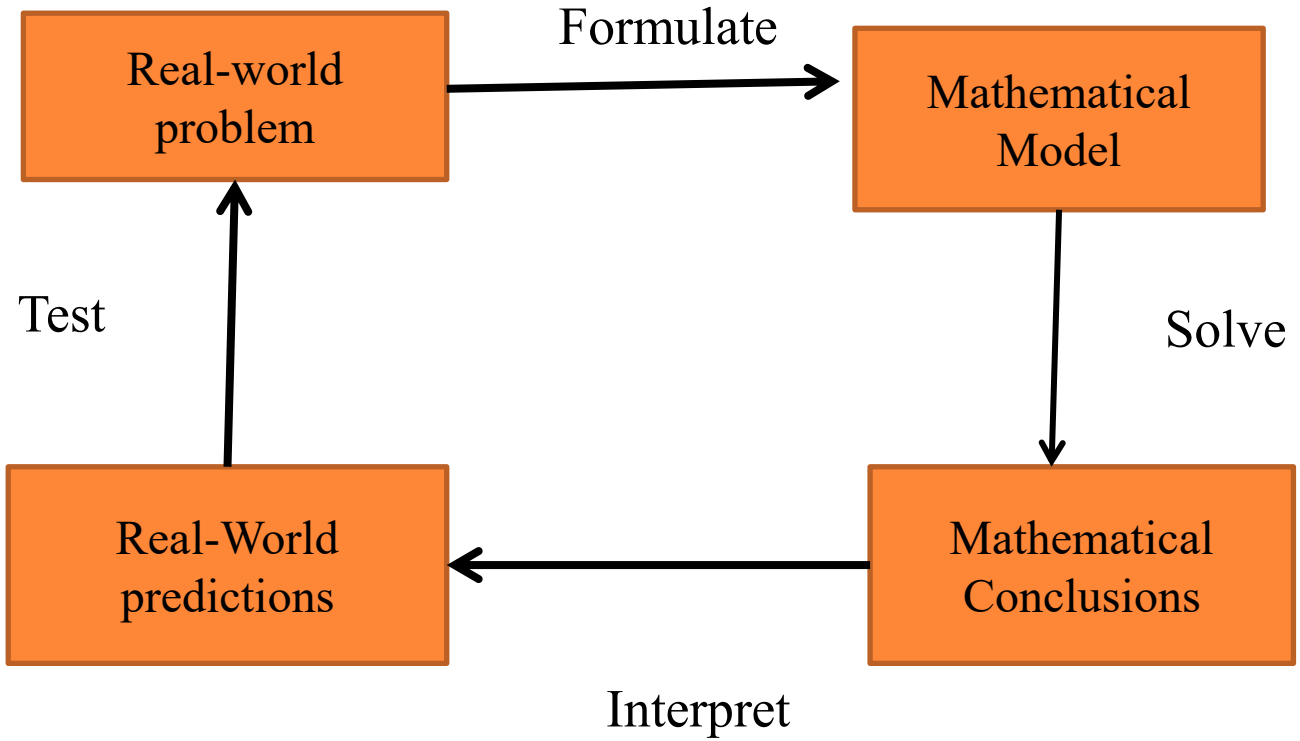
MATHEMATICAL MODELING

A mathematical model is a description or representation of a real-world phenomenon using mathematical concepts.

The purpose of the model is to understand the phenomenon

To make predictions about future behaviour

MATHEMATICAL MODELING



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MATHEMATICAL MODELING

Formulate: Expressing real-world problem in terms of Mathematical expression by

- Identifying and naming the independent and dependent variables.
- Making assumptions that simplify the phenomenon enough to make it mathematically tractable.
- Use knowledge of physical situation and Mathematical skills to obtain equations that relate the variables.

MATHEMATICAL MODELING

Solve: Apply the knowledge of mathematics that we know to solve the formulated problem to derive mathematical conclusions.

Interpret: From mathematical conclusions interpret the solution as information about the original real-world phenomenon by way of offering explanations or making predictions.

Test: Test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

The modeling error is associated with a possible inaccuracy of the mathematical model describing the real physical application at hand, and/or a possible inaccuracy in the data entering the model formulation (for example, due to measurement machine tolerances and/or statistical fluctuations of the phenomena under investigation).

MATHEMATICAL MODELING

Differential equations arise in many areas of

- science and technology
- specifically, whenever a deterministic relation involving some continuously varying quantities and their rates of change in space and/or time (expressed as derivatives) is known

MATHEMATICAL MODELING

Electrical Circuits: R-L circuit

Modelling electric circuits using differential equations is a common approach in electrical engineering and physics. The key to this modelling is to relate the voltage and current in circuit components using differential equations based on fundamental laws, such as Ohm's Law and Kirchhoff's Laws.

MATHEMATICAL MODELING

Ohm's Law states that the current flowing through a conductor between two points is directly proportional to the voltage across the two points and inversely proportional to the resistance between them. The mathematical expression for Ohm's law:

$$V = IR$$

Kirchhoff's Voltage Law: The voltage around a loop equals the sum of every voltage drop in the same loop for any closed network and equals zero.

$$L\frac{dI}{dt} + RI = E(t)$$

MATHEMATICAL MODELING

In standard form

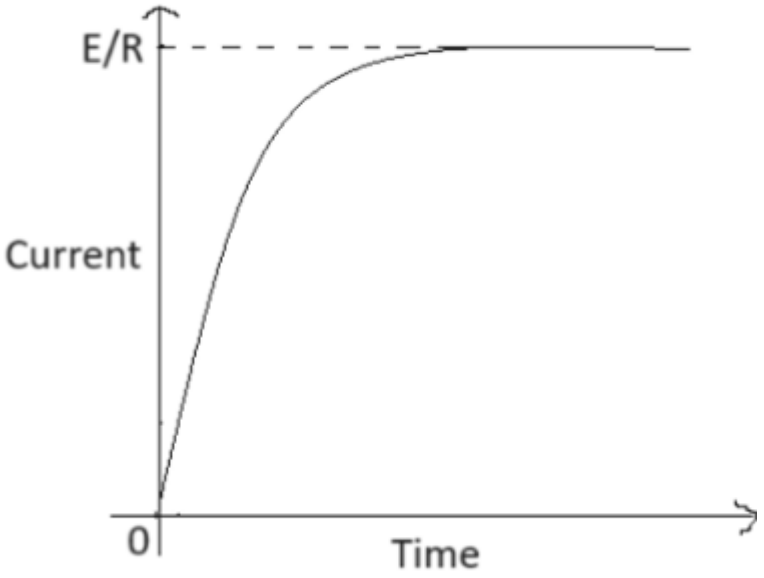
$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L}$$

On solving the above equation we get an expression for current *I(t)*, where *t* is time, as

$$\mathbf{I(t)} = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$$

MATHEMATICAL MODELING

- In general, at time t=0, the current I=0, i.e., there is no initial current.
- As *t* approaches infinity, the current I(t) approaches the value $\frac{E}{R}$.
- Notice that the current increases rapidly in the initial stage, followed by a slower rise in the later stage, eventually leveling off to $\frac{E}{R}$.
- The maximum current attained by the circuit is $\frac{E}{R}$



MATHEMATICAL MODELING

The Think Tank !!!

You are working with an L-R circuit connected to a battery. When you close the switch, the current begins to rise according to the expected behaviour of an L-R circuit. However, you decide to test the circuit by intermittently opening and closing the switch. Each time you open the switch, the current drops to zero instantaneously, and when you close the switch again, the current starts from zero instead of continuing from its previous value.

Can the given problem be modelled as a differential equation? Justify your answer.

Solution: No, the given problem cannot be modelled as a differential equation, since the circuit is tested by intermittently opening and closing the switch, giving rise to discontinuous current flow.

MATHEMATICAL MODELING

b. Specify the dependent variable and independent variable, in the case where the scenario can be modelled.

Solution: Current (I) is the dependent variable and time (t) is the independent variable.

c. When could this scenario be modelled accurately? Write the general form of the model.

Solution: The scenario could be modelled accurately, provided the current flow in the circuit was continuous. And, in this case, the scenario could be modelled as $\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$, where I is the current at any time t, R is the resistance and L is the inductance.

MATHEMATICAL MODELING

Population Model

Easiest mathematical model offered to govern the population dynamics of a certain species. One of the earliest attempts to model human population growth by means of mathematics

If *P(t)* denotes the total population at time *t*, then the rate at which population of a country grows at a certain time *t* is given by

$$\frac{dP}{dt} = kP(t) \text{ ---------- } -(i)$$

This model is with the assumption- rate at which a population of a country grows at a certain time is proportional to the total population of the country at that time.

MATHEMATICAL MODELING

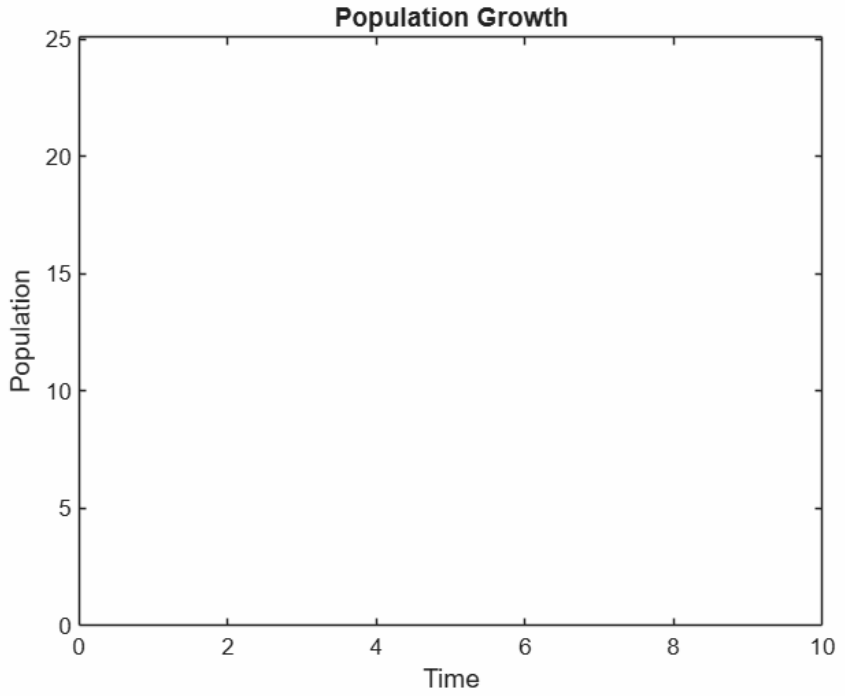
On solving we get, $\mathbf{P(t) = P_0 e^{kt}}$ - EXPONENTIAL MODEL

$\mathbf{P_0}$ is the initial population

k is called growth or decay const

If $k>0$?

If $k<0$?



MATHEMATICAL MODELING

The behaviour of the model depends on the value of the constant k:

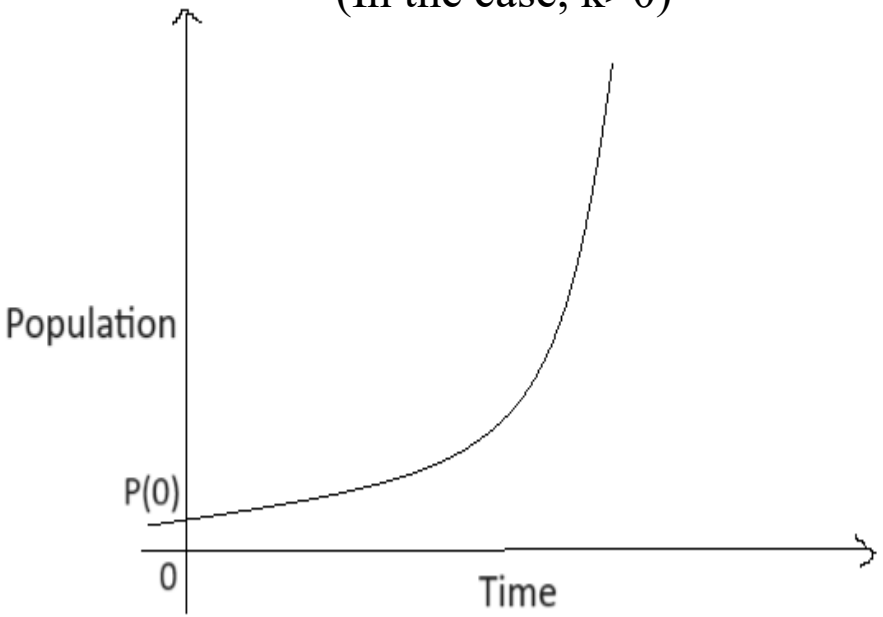
If $k > 0$: The model describes exponential growth. The population $P(t)$ increases over time as the exponential factor e^{kt} becomes larger. This is typical in scenarios where the growth rate is proportional to the current size of the population, such as in unchecked biological growth or compounding interest.

If $k < 0$: The model describes exponential decay. The population P(t) decreases over time as the exponential factor e^{kt} becomes smaller (since e^{kt} approaches zero as t increases). This is common in scenarios such as radioactive decay or depreciation of assets.

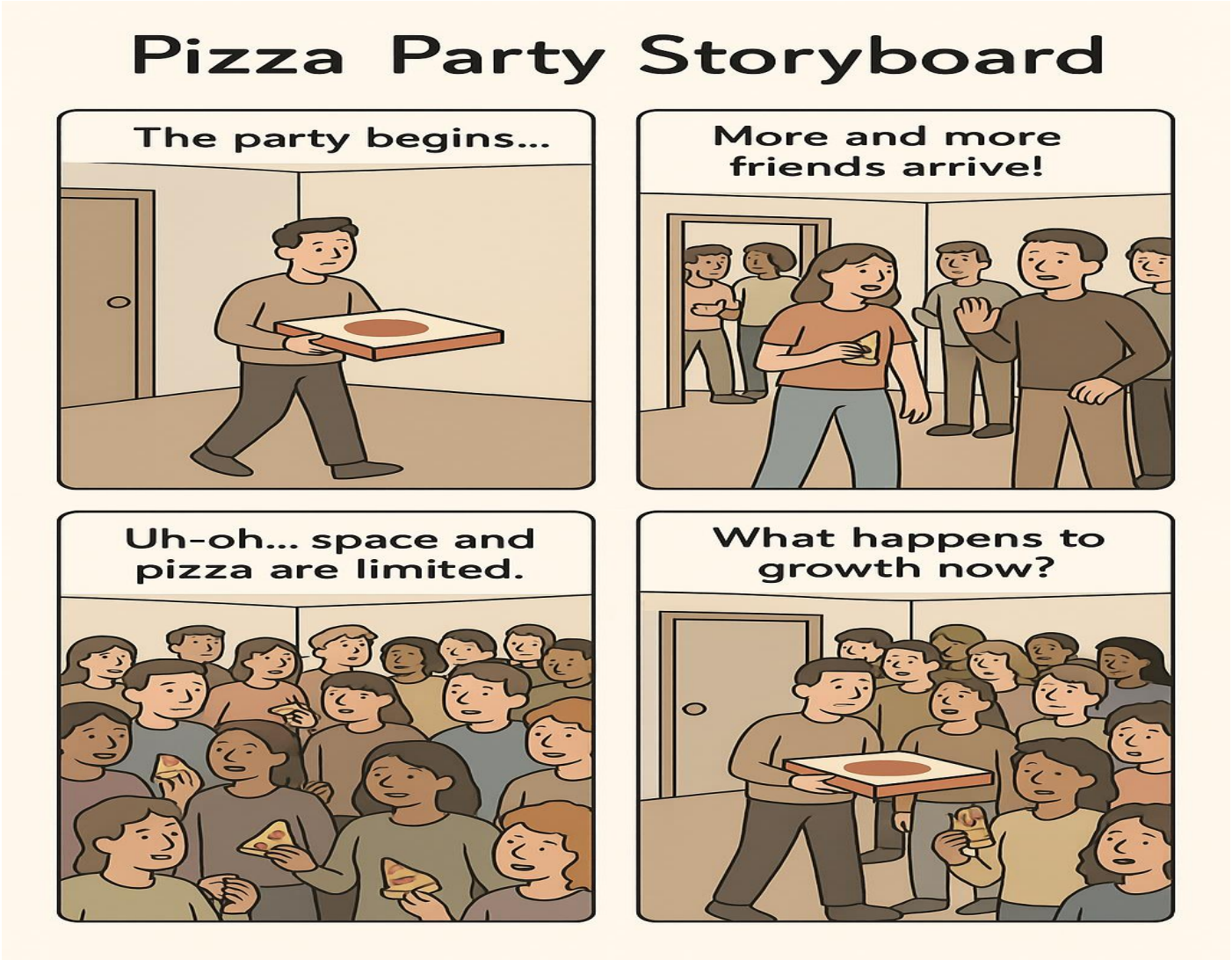
MATHEMATICAL MODELING

- At time $t = 0$, we have the initial population, $P(0)$.
- The factors influencing population growth include the growth constant (k), time (t), and the initial population $(P(0))$.
- An increase in any of the above mentioned parameters results in a corresponding increase in population.

(In the case, k>0)



MATHEMATICAL MODELING



MATHEMATICAL MODELING

This simple model which does not take many factors into account (immigration and emigration, occurrence of death in mass – corona virus, earthquake, floods for example) that can influence human populations to either grow or decline.

Never the less turned out to be fairly accurate in predicting the population.

Model fails when $t \to \infty$, why?

MATHEMATICAL MODELING

$$\lim_{t \to \infty} P(t) = \infty$$

The complication is that population growth is eventually limited by some factor

- one from among many - essential resources
- When a population is far from its limits of growth it can grow exponentially.
- However, when nearing its limits the population size can fluctuate, even chaotically.

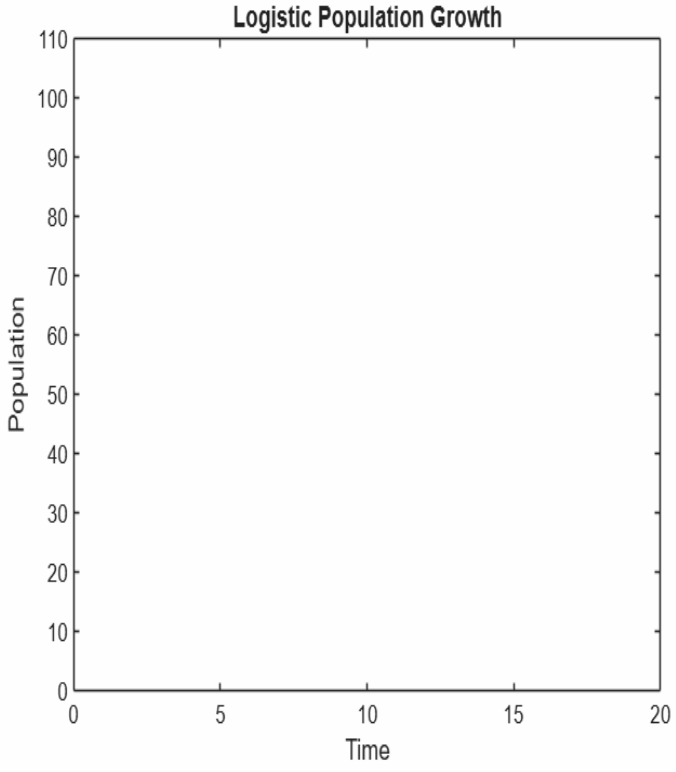
Another model was proposed to remedy this weakness in the exponential model.

It is called the **logistic model** (also called Verhulst-Pearl model).

MATHEMATICAL MODELING

Logistic Model

- Growth of a population is affected by birth rate, death rate, food supply, predators, etc.
- Growth constant **k** reflects net growth rate (births – deaths) per unit time.
- Does population growth rate stay constant or change over time?
- In many systems, population grows until it reaches a **steady state**.
- Exponential growth does not consider this limit.
- **Carrying capacity** explains that only a certain number of organisms can survive in a given area without resource shortages.

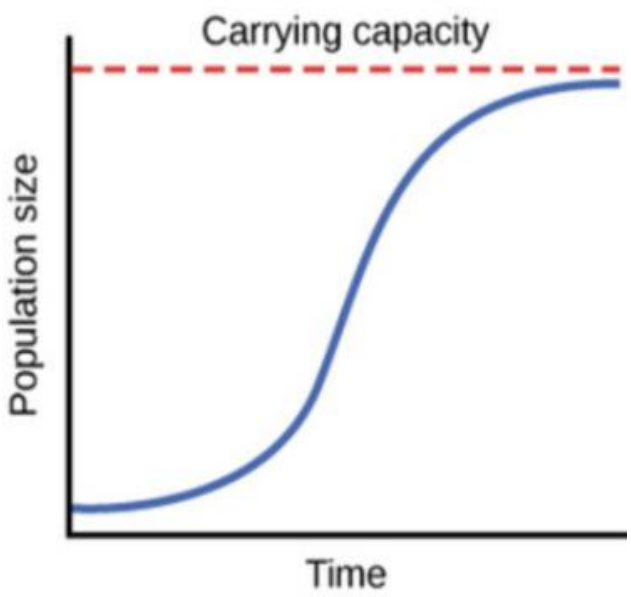


MATHEMATICAL MODELING

Let M represent the carrying capacity for a particular organism in a given environment, and let k be a real number that represents the growth rate. The function P(t) represents the population of this organism as a function of time t and the constant P_o represents the initial population (population of the organism at time t = 0). Assuming that, initially the growth rate is proportional to P, and P decreases if it ever exceeds M, the logistic differential equation is given by,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

This differential equation has a solution of the form: $P(t) = \frac{M}{1+\frac{M-P_0}{P_0}e^{-kt}}$.



MATHEMATICAL MODELING

The Think Tank !!!

1. Suppose you are studying the population growth of species of birds in a forest. You have collected data indicating that the intrinsic growth rate (k) is 0.2 per year, and the carrying capacity (M) of this species is 5000 birds. Consider the initial population to be 1000 birds.
Answer the following for the given situation:

a. Can the given problem be modelled as a differential equation? Justify your answer.

Solution: Yes, the given problem can be modelled as a differential equation, since the given scenario involves rate of change of population and the change in population is continuous. The scenario is modelled using logistic model, as $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, where $\frac{dP}{dt}$ represents the rate of change of population, P is the population at time t, M is the carrying capacity, k is the growth or decay constant.

MATHEMATICAL MODELING

b. Specify the dependent variable and independent variable.

Solution: Population (P) is the dependent variable and time (t) is the independent variable.

c. In the given problem what is the limiting factor for the bird population.

Solution: Carrying capacity is the limiting factor for the bird population.

MATHEMATICAL MODELING

Newton's Law of Cooling

- Newton's Law of Cooling finds applications in fields such as physics, engineering, and environmental science. It provides a fundamental understanding of how heat is exchanged between objects and their surroundings, aiding in the design and analysis of systems involving thermal processes.

- The law states that **'the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature'**. It is often used to model heat transfer processes.

MATHEMATICAL MODELING

Newton's Law of Cooling

Let T(t) represent the temperature of an object at time t, and let T_{env} be the ambient temperature. The differential equation for Newton's Law of Cooling can be expressed as follows:

$$\frac{dT}{dt} = -k(T(t) - T_{env})$$

where,

$\frac{dT}{dt}$ represents the rate of change of temperature with respect to time.

T(t) is the temperature of the object at time t.

T_{env} is the ambient temperature.

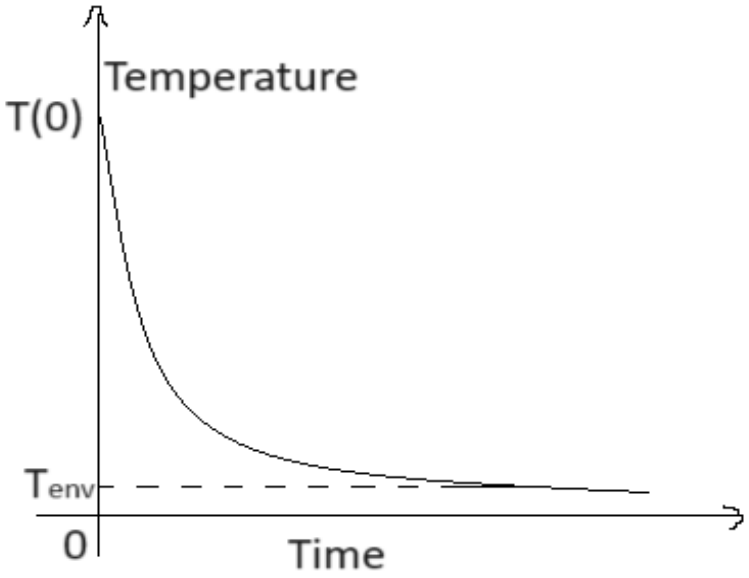
k is the cooling or heating constant, which depends on the specific properties of the object and the surrounding medium.

On solving the above equation, we get, $T(t) = T_{env} + Ce^{-kt}$.

MATHEMATICAL MODELING

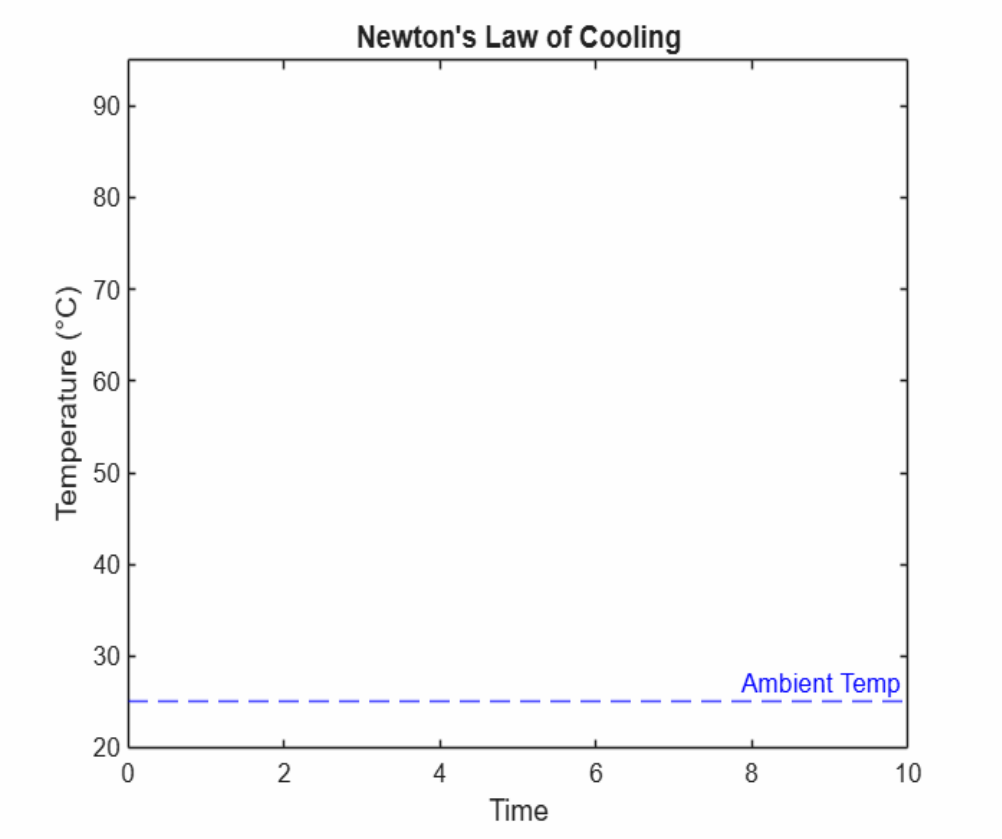
Newton's Law of Cooling

- At time t = 0, the object is at initial temperature.
- As the value of k increases, the temperature approaches T_{env}. The value of k is dependant on the characteristic of the object and the environment.
- With time, the temperature of the object approaches the environmental temperature, with the change occurring most rapidly during the initial stages.



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1. Which of the following best defines a mathematical model?

a) A real-world experiment
c) A physical representation of a problem
b) A description of a phenomenon using mathematical concepts
d) A graphical chart

Answer: b

2. In a population growth model, if P(t) represents population at time t, then P(t) is:

a) Independent variable
b) Dependent variable
c) Constant
d) Error term

Answer: b

3. Which of the following is an example of exponential model?

a) Radioactive decay
b) Oscillation of a pendulum
c) Projectile motion
d) Logistic population model

Answer: a

MATHEMATICAL MODELING

4. Which of the following real-world problems can be modeled by Newton's Law of Cooling?

a) Current in an L–R circuit
b) Cooling of hot coffee
c) Spread of a disease
d) Growth of bacteria in a lab

Answer: b

5. When a model prediction does not match real-world data, the difference is called:

a) Random error
b) Human error
c) Modeling error
d) Round-off error

Answer: c

MATHEMATICAL MODELING

The Think Tank !!!

1. You place a cup of hot coffee in a room to cool. However, the room's air conditioner malfunctions, causing abrupt temperature repeatedly. Additionally, the humidity in the room changes over time, causing condensation on the coffee cup's surface, affecting the heat transfer rate.

a. Can the given problem be modelled as a differential equation? Justify your answer.

Solution: No, the given problem cannot be modelled as a differential equation, since there is abrupt change (more fluctuation) in temperature.

MATHEMATICAL MODELING

b. When could this scenario be modelled accurately? Given $T(0) = 85°C$, $T_{env} = 20°C$, and $k = 0.03\ min^{-1}$. Write the differential equation governing the temperature of the coffee. State the initial condition.

Solution: The scenario could be modelled accurately, provided the temperature change was continuous and not abrupt. And, in this case, the scenario could be modelled using Newton's law of cooling,

$\boldsymbol{\frac{dT}{dt} = -k(T - T_{env})}$, where $\frac{dT}{dt}$ represents the rate of change of temperature, T is the temperature of the object at time t, T_{env} is the ambient temperature, k is the cooling or heating constant.

c. Specify the dependent variable and independent variable, in the case where the scenario can be modelled.

Solution: Temperature (T) is the dependent variable and time (t) is the independent variable.

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Differential Equations:

A differential equation is an equation which involves differential coefficients or differentials.

- $\frac{dy}{dx} = x^2 - 1$
- $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$
- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0$
- $y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}}$
- $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

Differential Equations - Types & Terminologies

- An **Ordinary Differential Equation (ODE)** is that in which all the differential coefficients have reference to a single independent variable.

- A **Partial Differential Equation** is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.

- The **order of a differential equation** is the order of the highest derivative appearing in it.

- The **degree of a differential equation** is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Differential Equations – Level 1 Questions (CO1)

- Identify the order and degree of the following differential equations:

1) $\frac{dy}{dx} = x^2 - 1$

2) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$

3) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0$

4) $y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}}$

5) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

Note:
A differential equation is said to be **linear,** if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

Answers:

a) Order: First; Degree: First

b) Order: Third; Degree: First

c) Order: First; Degree: First

d) $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$; Order: First; Degree: Second

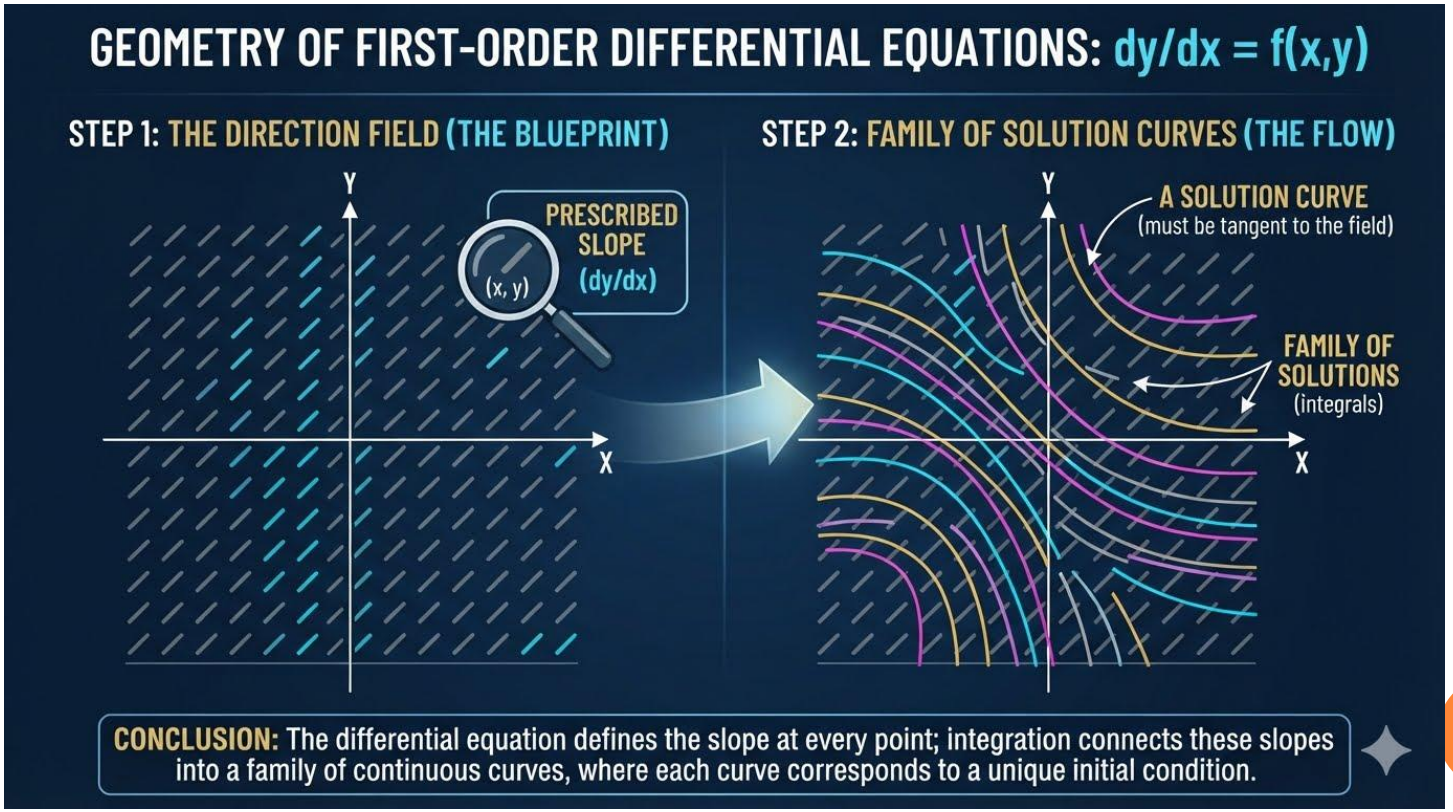
e) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$; Order: Second; Degree: Second

Geometrical meaning of a first-order first-degree D.E.:

A first-order first-degree differential equation is generally written as

$\frac{dy}{dx} = f(x, y)$.

The derivative $\frac{dy}{dx}$ represents the slope of the tangent to the curve at any point (x, y).



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Geometrical meaning…

Hence:

- **The differential equation determines the slope at every point.**
- **A family of curves satisfying the equation can be obtained.**
- **The collection of slopes forms a direction field.**
- **The solution curve passes through points while maintaining the specified slope.**

Thus, geometrically, a first-order differential equation represents a **family of curves whose tangent at each point has a prescribed slope.**

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Formation of Differential Equations

Differential equations are formed by elimination of arbitrary constants.

- **Number of constants = Order of the derivative.**
- One constant (C) requires a **first-order** derivative.
- Two constants (C_1, C_2) require a **second-order** derivative.
- n constants $(C_1, C_2, \ldots\ C_n)$ require an $\boldsymbol{n^{th}}$ **order** derivative.

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Formation of Differential Equations …

Step-by-Step Process

1. **Write the given equation** containing the constants.
2. **Count the constants** to know your target derivative order.
3. **Differentiate** the equation with respect to (x) that many times.
4. **Eliminate** the constants using your original equation and your new derivative equations.
5. **Result:** A clean differential equation with no constants left!

Formation of Differential Equations …

Level 1 Questions (CO1)

- **Form the differential equation by eliminating the arbitrary constant from $y = A\,e^{2x}$.**

Ans:

Given: $y = A\,e^{2x}$

Differentiate both sides with respect to x: $\frac{dy}{dx} = 2A\,e^{2x}$

Since $A\,e^{2x} = y$

we get $\frac{dy}{dx} = 2y$

Therefore, the required differential equation is: $\frac{dy}{dx} - 2y = 0$

Level 1 Questions (CO1) …

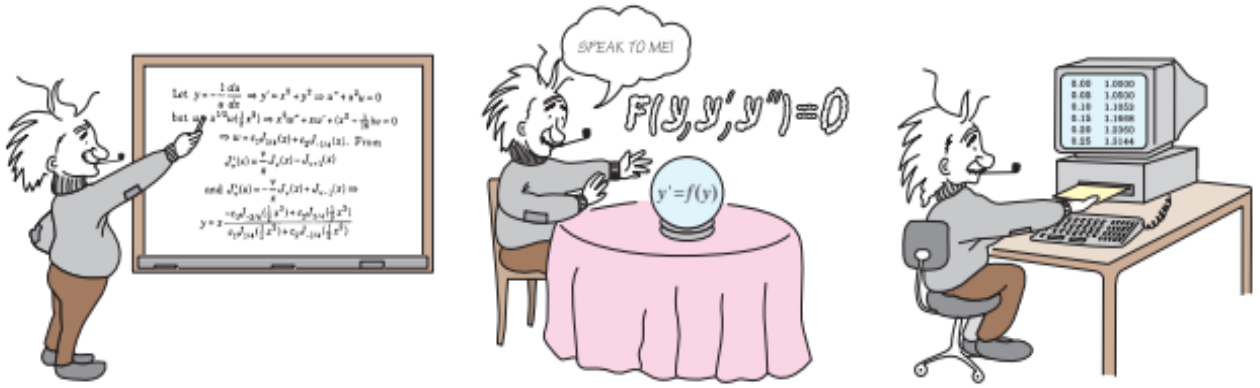
Eliminate the arbitrary constants and obtain the differential equations:

1. $y = cx + c^2$
2. $y = A + Bx + Cx^2$
3. $y = A\cos 2t + B\sin 2t$
4. $y = A\,e^{3x} + B\,e^{2x}$
5. $y = A\,e^{x} + B\,e^{-x} + C$
6. $y = ax^3 + bx^2$
7. $y^2 - 2ay + x^2 = a^2$

Different Approaches to the study of Differential Equations

Approaches to solve a Differential Equation

- Analytical Methods
- Numerical Methods



Consider the first order IVP (Initial Value Problem) of the form,

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

1. Under what conditions, there exists a solution to (1)?
2. Under what conditions, there exists a unique solution to (1)?

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THE FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM FOR FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Suppose $D = \{(x, y) | a < x < b \text{ and } c < y < d\}$ is an open rectangle, that contains the point (x_0, y_0) and that the function $f(x, y)$ is continuous on D. If f satisfies Lipschitz condition on D in the variable y (i.e., if a constant $L > 0$ exists with $|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|$, whenever (x, y_1) and (x, y_2) are in D), then the initial-value problem (1) has a **unique solution** on some open sub-interval of (a, b) which contains the point x_o.

1. Consider the IVP:

$$y' = x - y + 1, \qquad y(1) = 2.$$

Here, $f(x, y) = x - y + 1$.
Clearly, the function f is continuous .
For any y_1, y_2 in R, we have,

$$f(x, y_1) - f(x, y_2) = -(y_1 - y_2) .$$

i.e., $|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|$, with $L = 1$.
Hence, the fundamental existence and uniqueness theorem ensures that in some open interval centered at 1, the solution of the given IVP exists.

2. Consider the IVP:

$$y' = y^{1/3} + \mathrm{x}, \qquad \mathrm{y}(1) = 0$$

Here, $f(x, y) = y^{1/3} + x$.

Clearly, the function f is continuous.

We calculate the difference:

$$|f(x, y_1) - f(x, y_2)| = \left| {y_1}^{\frac{1}{3}} - {y_2}^{\frac{1}{3}} \right|.$$

However, near y=0, this difference can become arbitrarily large compared to $|y_1 - y_2|$, because the derivative of $y^{1/3}$ becomes infinite as y→0. Thus, f does not satisfy the Lipschitz condition near $y = 0$.

Hence, the fundamental existence and uniqueness theorem cannot ensure uniqueness of the solution.

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Types Of Ordinary Differential Equation

- **Variable Separable DE**
- Homogeneous DE
- **Linear DE**
- Bernoulli's DE
- Exact DE

Differential Equation and Solution Of Differential Equation

- General Solution
- Particular Solution

VARIABLE SEPARABLE METHOD

Suppose the given differential equation is in the form $\frac{dy}{dx} = f(x)g(y)$

By separating the variables, it can be written in the form

$$f(x)\,dx = \frac{dy}{g(y)}$$

By integrating the LHS with respect to x and RHS with respect to, the required solution will be obtained.

Example : Solve $\frac{dy}{dx} = xy$

Solution: Let us rearrange the given DE as $\frac{dy}{y} = x dx$

On integration, we have $\int \frac{dy}{y} = \int x\,dx$

$$\log y = \frac{x^2}{2} + C$$

is the required solution.

LINEAR DIFFERENTIAL EQUATIONS

A **linear differential equation** is a differential equation in which the dependent variable y and all its derivatives appear **only to the first power** and are **not multiplied together**.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \quad \ldots\ldots\ldots\ldots(1)$$

Where P and Q are functions of x only or may be constants.

Solution of (1) is $ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C$

LINEAR DIFFERENTIAL EQUATIONS

Example: Solve $\frac{dy}{dx} + \frac{y}{x} = x^3$

Solution: The given differential equation is in the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{1}{x}, Q = x^3$

I.F $= e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{logx} = x$

The solution is given by $y(I \cdot F) = \int Q(I \cdot F)\,dx + C$

$\Rightarrow yx = \int x^3 x\,dx + C$

$\Rightarrow yx = \frac{x^5}{5} + C$

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PROBLEMS:

Solve the following ODEs

i) $\frac{dy}{dx} + 3x^2y = 6x^2$ Solution : $y = 2 + Ce^{-x^3}$

ii) $\frac{dy}{dx} = x^2e^{-y}$ Solution : $e^y = \frac{x^3}{3} + C$

iii) $\frac{dy}{dx} = (1 + y^2)x$ Solution : $\tan^{-1} y = \frac{x^2}{2} + C$

iv) $x\,dy - (y + 2x^2)\,dx = 0$ Solution : $\frac{y}{x} = 2x + c$

v) $\frac{dy}{dx} + 2xy = 6e^{-x^2}$ Solution : $ye^{x^2} = 6x + C$

vi) $\frac{dy}{dt} = 3y$ Solution : $y = Ce^{3x}$

PROBLEMS:

Solve the initial-value problem

i) $\frac{dy}{dx} = -\left(\frac{x}{y}\right),\ y(4) = -3$

Solution : $\int y\,dy = -\int x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

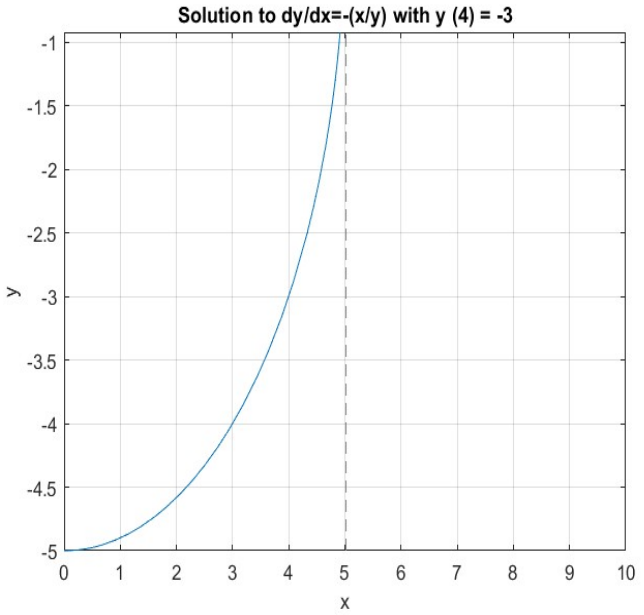
$$y^2 + x^2 = c$$

$$y(4) = -3$$

x = 4, y = 3

$-3^2 + 4^2 = c$

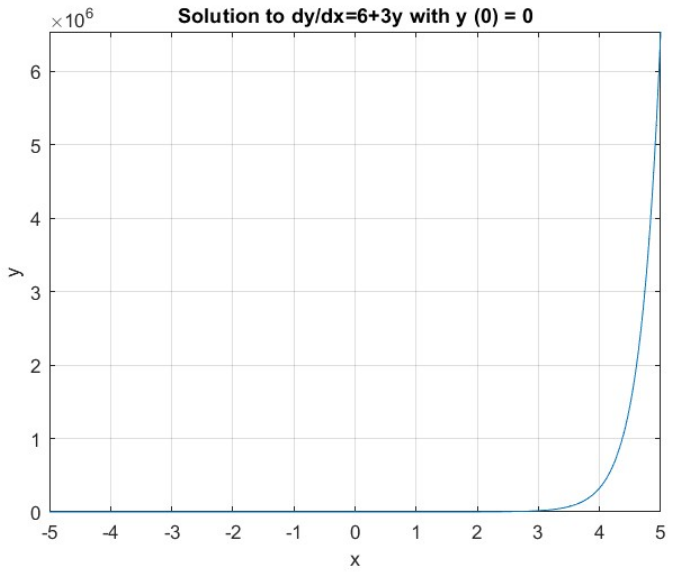
For the given initial conditions, c = 25



ii) $\frac{dy}{dx} - 3y = 6,\ y(0) = 0$

Solution: $y = -2 + Ce^{3x}$

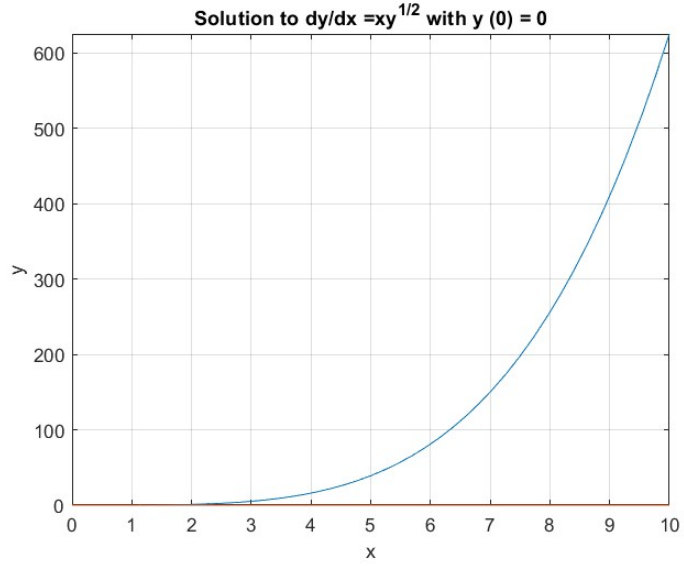
For the given initial condition, c = 2



iii) $\frac{dy}{dx} = xy^{\frac{1}{2}}$, $y(0) = 0$

Solution: $y = \frac{x^4}{16} + C$

For the given initial condition, c = 0



QUIZ

1. In the Variable Separable method, a first-order DE can be written as:

a) $M(x,y)dx + N(x,y)dy = 0$

b) $f(x)dx + g(y)dy = 0$

c) $dy = f(x)dx$

d) None of the above

Answer: b & c

2. A Linear Differential Equation has the form:

a) $\frac{dy}{dx} + 3y = 5x$

b) $\frac{dy}{dx} = y^2$

c) $y' + y = 0$

d) $\frac{dy}{dx} = e^x$

Answer: a & c

- Suppose you are a biologist studying the population dynamics of fish in a lake. Your goal is to model how the fish population changes over time, taking into account factors such as reproduction, predation and environmental conditions.

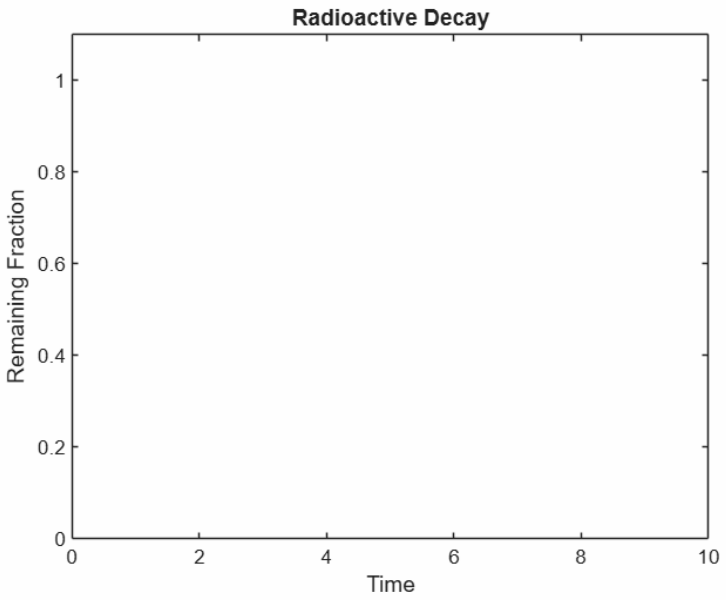
- A constant electromotive force E volt is applied to a circuit containing constant resistance R ohms in series and constant inductance L henries. Can you find how the current changes in the circuit.

- Suppose you have a freshly baked pizza that has just come out of the oven. You want to know how long it will take for the pizza to cool down to a safe eating temperature.

Can The above Examples Be Modeled As a Differential Equation?

The Examples explained above are real world scenarios of

- Growth and Decay
- LR Circuit
- Newton's Law Of Cooling



which are modeled as First-Order
and First-Degree Differential Equations
and are solved using the following methods

- Variable Separable
- Linear Differential Equation

PROBLEMS:

1. Rabbit Population Dynamics:

Suppose you have a population of rabbits on an isolated island. The population of rabbits can be described as follows. Rabbits reproduce at a constant rate of 0.2 per year per rabbit. Rabbits experiences a death rate that is proportional to their current population, with a constant proportionality of 0.1 per year. At the start, you have 100 adult rabbits on the island. Find the rabbit population dynamics over time.

Solution: k = Birth rate – Death rate = 0.2 – 0.1 = 0.1

Population at time t, $\frac{dP}{dt} = kP(t)$

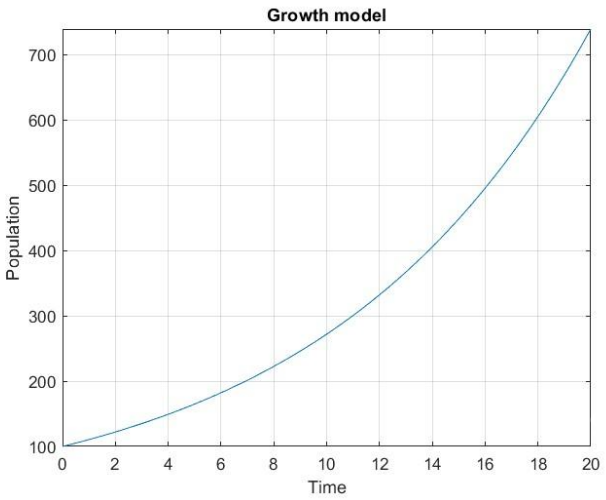
$$\int \frac{dP}{dt} = \int 0.1\, dt$$

$$P = Ce^{0.1t}$$

Population of the rabbits at the start is 100 on the island i.e. when t = 0, P = 100.

Hence the particular solution is

$$P = 100e^{0.1t}$$



PROBLEMS:

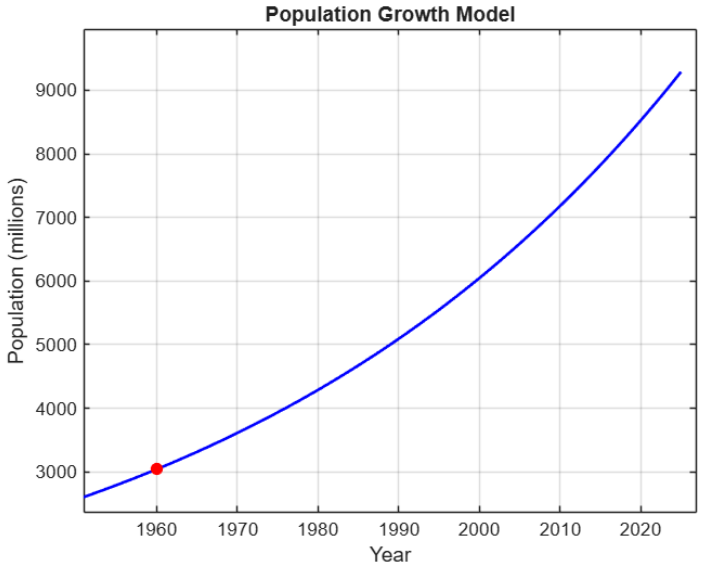
2. The world population in 1950 was about 2,560 million, and by 1960 it had grown to about 3,040 million. Assuming that the population follows an exponential growth model, determine the relative growth rate and use this model to estimate the population in 1993 as well as to predict the population in 2020.

Solution: $P = Po\mathrm{e}^{kt}$

k = 0.0171

Population in 1993 = 5360 million

Population in 2020 = 8524 million



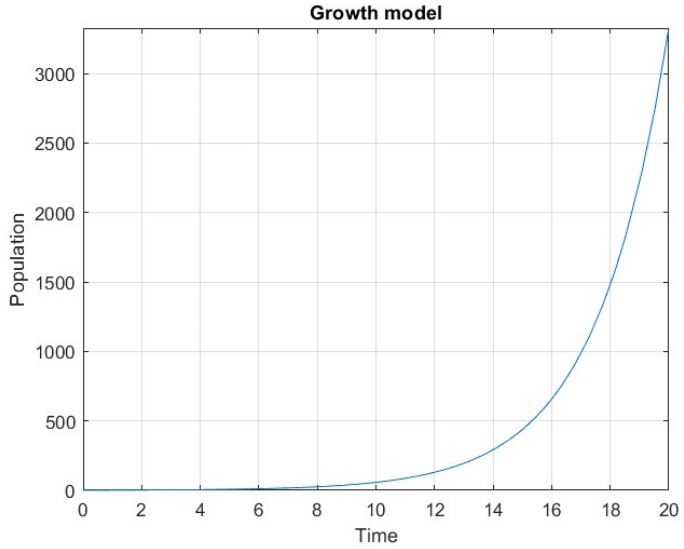
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PROBLEMS:

3. A culture initially has P_o number of bacteria. At t = 1 h the number of bacteria is measured to be $\frac{3}{2} P_o$. If the rate of growth is proportional to the number of bacteria P(t) present at time t, determine the time necessary for the number of bacteria to triple.

Solution: Time taken for the number of bacteria to triple is 2.7127 h

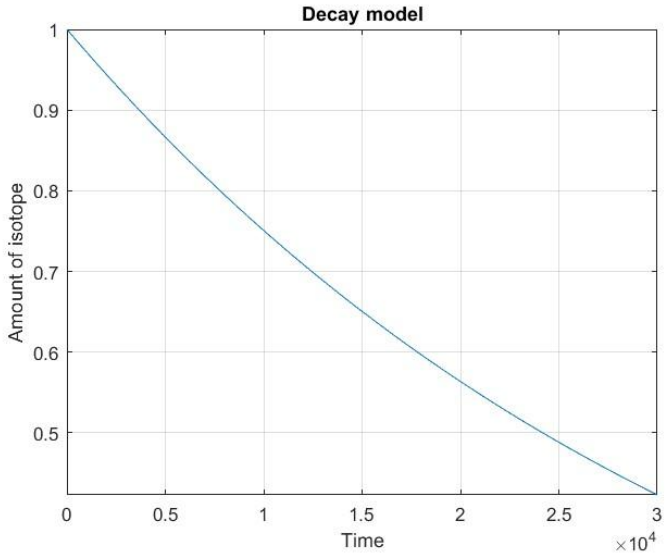
= 2 h 42.762 min



PROBLEMS:

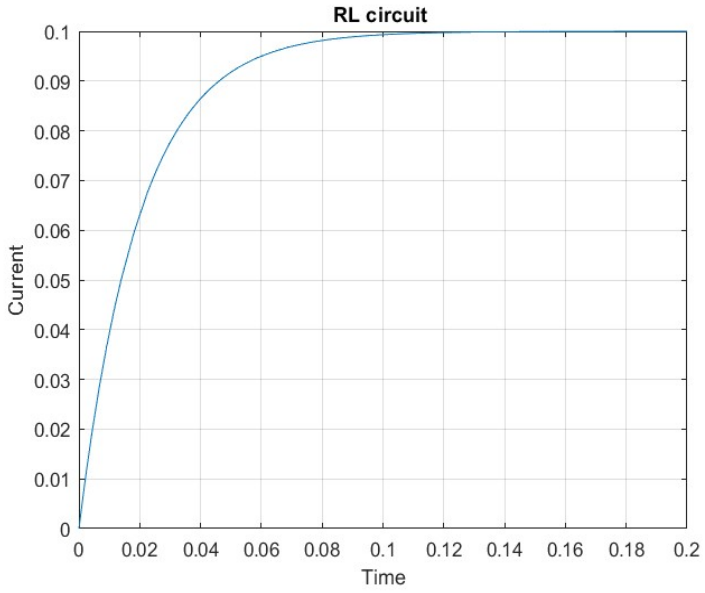
4. A breeder reactor converts relatively stable uranium-238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution: Half-life of this isotope if the rate of disintegration is proportional to the amount remaining is 24,173.4096 years.



PROBLEMS:

5. An RL circuit has an EMF of 5 V, a resistance of 50 Ω, an inductance of 1 H, and no initial current. Find the current in the circuit at any time *t*. What is the limiting value of the current.



Solution: $\frac{dI}{dt} + \frac{RI}{L} = \frac{E(t)}{L}$

On solving the above equation we get

$$I = \frac{E(t)}{\mathrm{R}} + C\mathrm{e}^{\frac{-R}{L}t} \Rightarrow I = \frac{5}{50} + C\mathrm{e}^{\frac{-50}{1}t}$$

C = 0.1

The current in the circuit at any time *t is* $i = 0.1(1 - e^{-50t})\ A$.

Limiting value of the current i.e when $t = \infty,\ i = 0.1\ A$

PROBLEMS:

5. Suppose that in winter the day time temperature in a certain office building is maintained at $70^0\ F$. The heating is shut off at 10 PM and turned on again at 6 AM. On a certain day the temperature inside the building at 2AM was found to be $65^0\ F$. The outside temperature was 50^0 F at 10PM and had dropped to 40^0 F by 6AM. What was the temperature inside the building when the heat was turned on at 6AM?

Solution: $\frac{dT}{dt} = -k(T(t) - T_{env})$

T_{env} = Avg(T(10PM)& T(6PM)) = 45 ^0F

On solving the equation, we get

$T(t) = \left(Ce^{-kt} + T_{env}\right)$

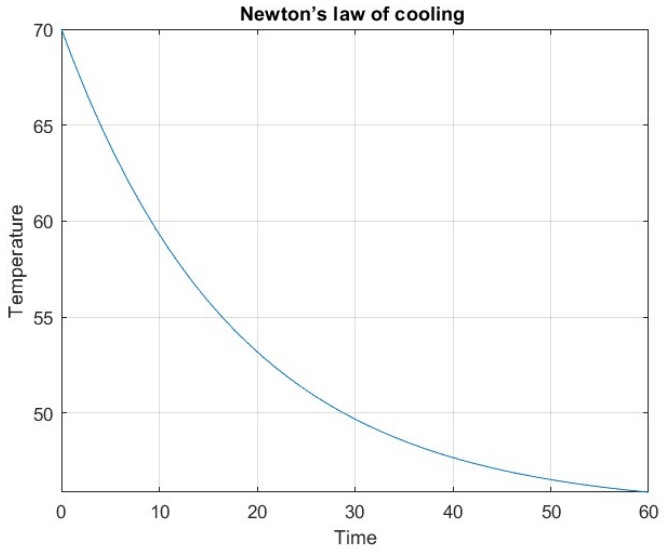
When t=0, T(t) = 70 ^0F

$\Rightarrow$ C = 25 ^{0} F

When t=4, T(t) = 65 ^0F

$\Rightarrow$ k = 0.0558

$T(t) = 45 + 25e^{-0.558t}$. Temperature inside the building when the heat was turned on at 6AM, $T(t) = 60.99^0F$



PROBLEMS:

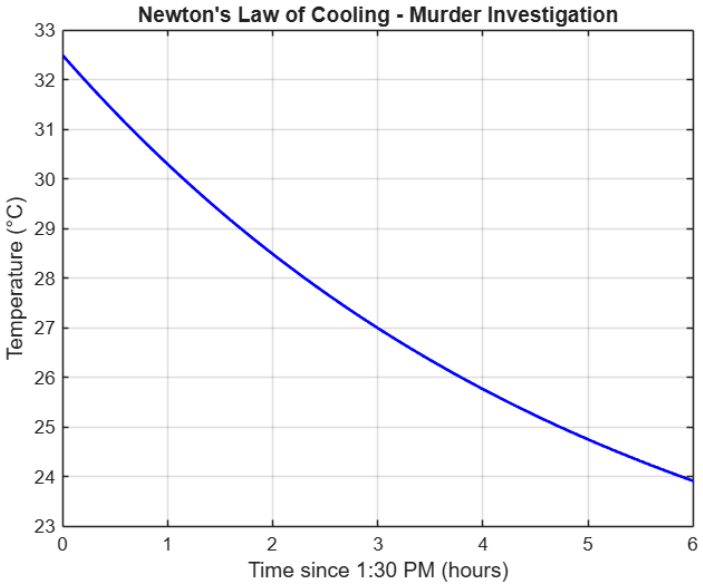
6. In a murder investigation, the temperature of the corpse was 32.5^0C at 1.30 PM and 30.3^0C an hour later. Normal body temperature is 37.0^0C and the temperature of the surroundings was 20.0^0C. When did the murder take place ?

Solution: General solution,

$T(t) = 20 + (37 - 20)\mathrm{e}^{-kt}$.

$\mathrm{k} = 0.193$

The murder took place at approximately **11:54 AM**.



LOGISTIC MODEL PROBLEMS

1. A population of rabbits on a small island is observed to grow according to the logistic model due to limited resources and space. Initially, there are 100 rabbits, and the island's carrying capacity is estimated at 1000 rabbits. The intrinsic growth rate of the population is 0.3 per year. However, this growth rate fluctuates annually due to environmental changes, which can be represented by the function 0.5 sin(t), where t is time in years.

The logistic growth equation with a fluctuating growth rate can be expressed as:

$$\frac{\mathbf{d}\boldsymbol{P}}{\mathbf{d}\boldsymbol{t}} = \boldsymbol{kP}\left(\mathbf{1} - \frac{\boldsymbol{P}}{\boldsymbol{M}}\right)$$

where, P is the population size, k = 0.3 + 0.5 $sin(t)$ is the time-varying growth rate, M = 1000 is the carrying capacity. Thus, we have

$$\frac{dP}{dt} = \left(0.3 + 0.5\, sin(t)\right)P\left(1 - \frac{P}{1000}\right)$$

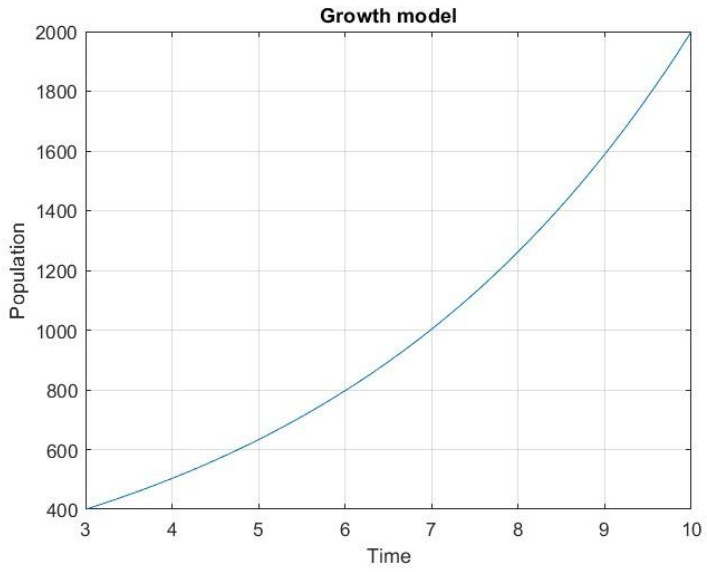
2) A population of deer in a national park is modeled by the logistic growth equation. Initially, there are 200 deer, and the park can sustainably support a maximum of 1500 deer. The intrinsic growth rate of the deer population is 0.2 per year. However, due to periodic droughts, the growth rate fluctuates annually according to the function 0.03 sin(t).

3) A population of fish in a newly created lake is modeled by the logistic growth equation. Initially, there are 500 fish, and the lake's ecosystem can support a maximum of 10,000 fish. The intrinsic growth rate of the fish population is 0.15 per year. However, due to seasonal variations in water temperature and food availability, the growth rate fluctuates annually according to the function $0.02 \sin(2\pi t)$, where t is time in years.

PRACTICE PROBLEMS

1. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t = 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

Solution: The initial number of bacteria $P_0 = 200.6906$



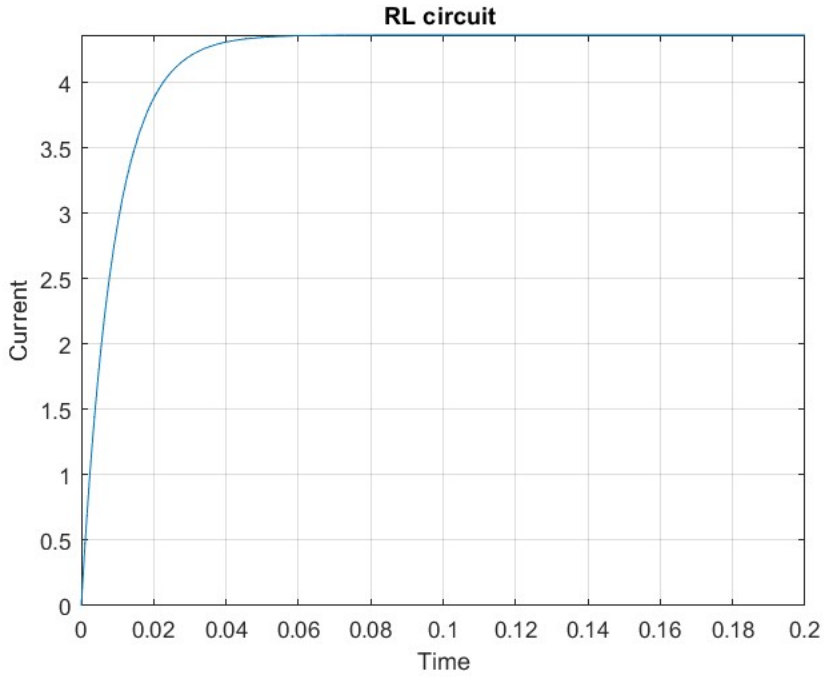
PRACTICE PROBLEMS

2. Solve the resulting ODE for the current $I(t)\ A$, where t is time. Assume that the circuit contains an EMF $E(t) = 48\ V$, a resistor $11\ ohms$, an inductor $0.1\ H$, and that the current is initially zero. When does the current reaches its steady state.

Solution: General solution,

$I(t) = \frac{48}{11}(1 - e^{-110t})$,

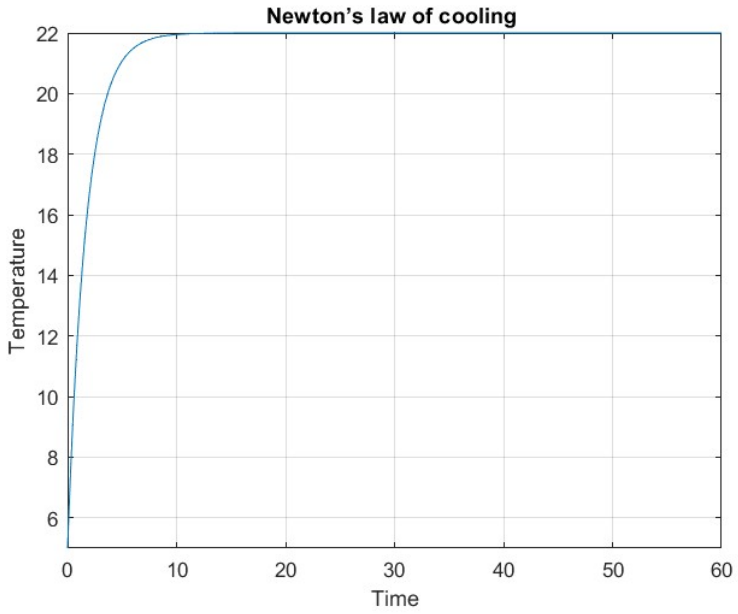
$I(\infty) = 4.3636$ A



PRACTICE PROBLEMS

3. A thermometer reading 5^0C is brought into a room whose temperature is 22^0C. One minutes later the thermometer reading is 12^0C. How long does it take until the reading is practically 22^0C.

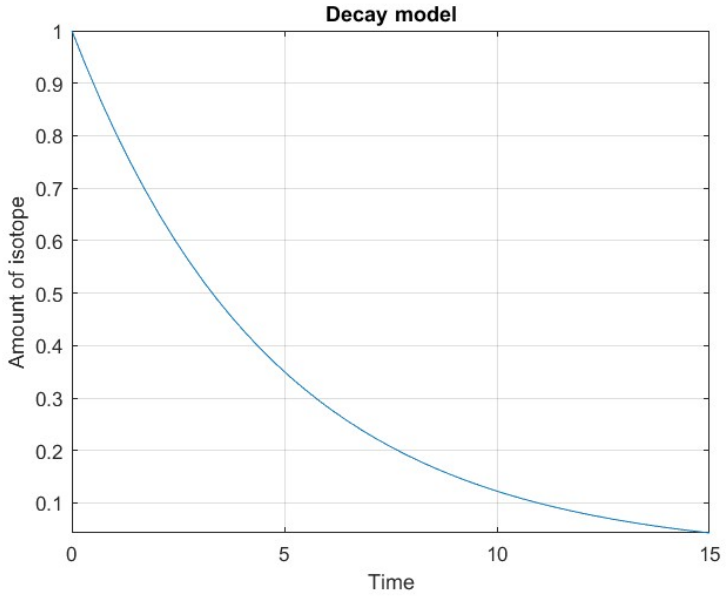
Solution: The reading is practically 22^0C when $t \to \infty$.



PRACTICE PROBLEMS

4. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

Solution: Time taken for 90% of the lead to decay is 11h (10.97 h).

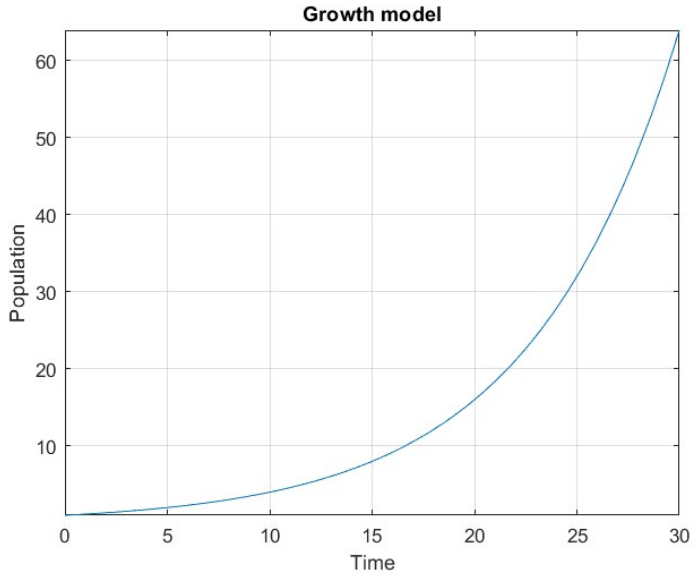


HOMEWORK PROBLEMS

1. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

   Solution: Time taken for the population to triple is 7.9264 years ;

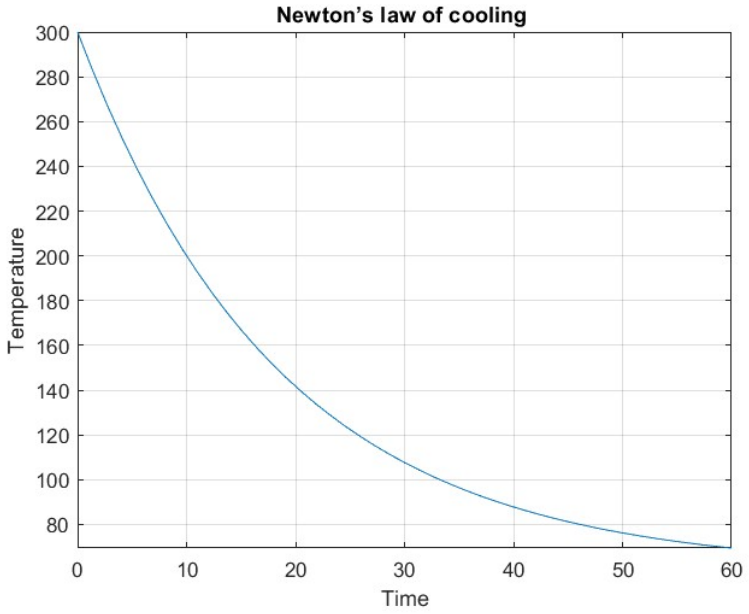
   Time taken for the population to triple is 10 years.



HOMEWORK PROBLEMS

2. If the temperature of the cake is 300^0F when it leaves the oven and is 200^0F ten minutes later, when will it be practically equal to the room temperature of 60^0F.

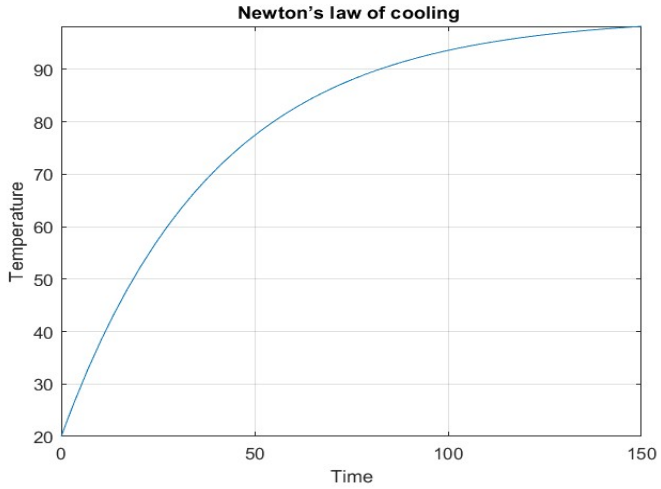
Solution: Cake will be practically equal to the room temperature of 60^0 when $t \to \infty$.



HOMEWORK PROBLEMS

3. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° in 1 second? How long will it take the bar to reach 98° C?

Solution: Time taken for the bar to reach 90° C is 82.1913 seconds and time taken for the bar to reach 98° C is 145.8055 seconds .



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QUIZ

1. The exponential population growth model assumes:
   a) Growth rate is constant and limited
   b) Growth rate is proportional to current population
   c) Growth depends only on food availability
   d) Growth follows a logistic curve

   Answer: b

2. In the logistic model, the term carrying capacity (M) refers to:
   a) The growth rate of the population
   b) The maximum sustainable population
   c) The doubling time of population
   d) A constant rate of decay

   Answer: b

3. If $k < 0$ in the exponential model $P(t) = P_0 e^{kt}$, the population:
   a) Grows indefinitely
   b) Remains constant
   c) Decays over time
   d) Oscillates periodically

   Answer: c

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QUIZ

4. Which of the following situations can be modeled by the logistic model?
   a) Spread of rumors in a closed group
   b) Motion of a freely falling object
   c) Heating of water in a kettle
   d) Decay of radioactive isotopes

Answer: a

5. A cup of coffee cools faster in:
   a) A warm room
   b) A room with the same temperature as coffee
   c) A cold room
   d) Room temperature does not matter

Answer: c

- Solve $(y^2 - x^2)\,dx = (y^2 + x^2)\,dy$

- Solve $\frac{dy}{dx} = x^2 + y^2$ and y(1) = 1.5

When analytical methods fail due to the inherent complexities of the equations or the specific conditions of the problem, **numerical methods**, in particular, are used for solving such equations because they can provide accurate solutions.

Numerical Solutions To First order and First Degree Differential Equations

- Euler's and Modified Euler's Method
- Runge -Kutta method

ERRORS AND APPROXIMATION

Error Analysis: The numerical error is the difference between the exact solution and the approximated solution .

Types of error:

- **Inherent Error:** Inherent error exist in the problem either due to approximate given data, limitation of the computing aids such as mathematical tales, desk calculator etc, Such type of error can be reduced by taking better data and using precision computing aids .

- **Round off error:** These error are due to rounding off the number

  Ex: $1.24372 \cong 1.244$

  Error $1.244 - 1.24372 = 0.00028$

ERRORS AND APPROXIMATION

- **Truncation Error: T**hese error arise due to use of truncating the infinite series to some approximate terms.

  Ex: Exponential series, Sin x, etc.

- **Absolute Error:** Absolute Error is the positive difference between the actual value and the approximate value.

  Absolute Error = actual value ~ approximation

- **Relative Error:** $\frac{absolute\ error}{actual\ value}$

- **Percentage Error:** $\frac{absolute\ error}{actual\ value} * 100$

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ORDER OF CONVERGENCE

- The order of convergence is a concept used in numerical analysis to describe how quickly an iterative numerical method approaches the desired solution. It is often expressed as a rate at which the error decreases with each iteration. There are several different orders of convergence, including linear, quadratic, cubic, and so on.

- Higher orders of convergence generally indicate faster convergence, which means the numerical method approaches the true solution more quickly with each iteration. However, achieving higher orders of convergence may require additional computations or more complex algorithms.

EULER'S & MODIFIED EULER'S METHOD:

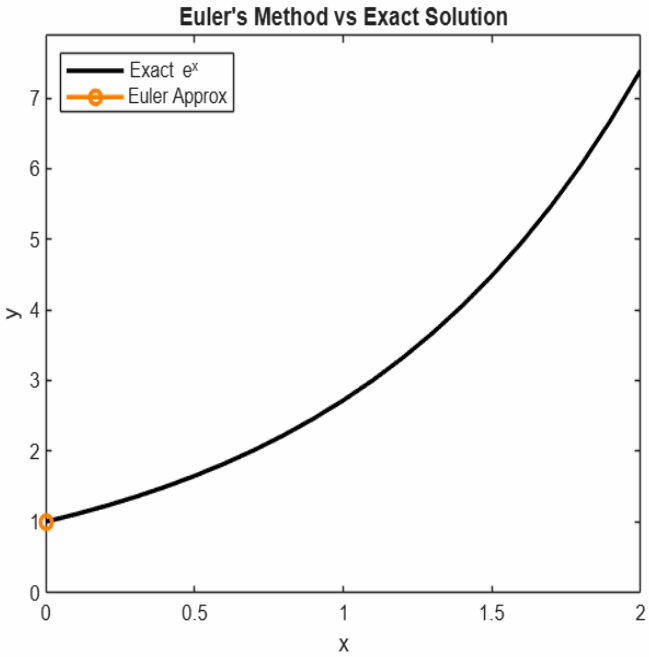
An Initial Value Problem of this equation is of the form $y_1 = f(x, y), y(x_0) = y_0$. The initial approximation for y_1 at $x_1 = x_0 + h$ is found using **Euler's** formula, which is given by $y_1^{(0)} = y_0 + h\left(f(x_0, y_0)\right)$. The other approximations are found using **Modified Euler's** formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right)$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right)$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right)$$

And so on.



RUNGE-KUTTA METHODS:

An Initial Value Problem of this equation is of the form

$$y_1 = f(x, y), y(x_0) = y_0$$

To find y_1 at $x_1 = x_0 + h$ we use the following formulae of **R-K method**

$$k_1 = h\left(f(x_0, y_0)\right)$$

$$k_2 = h\left(f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\right)$$

$$k_3 = h\left(f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\right)$$

$$k_4 = h\left(f(x_0 + h, y_0 + k_3)\right)$$

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

MATHEMATICAL MODELING

The Think Tank !!!

1. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° C in 1 second? How long will it take the bar to reach 98° C?

Answer the following for the given situation:

a. Determine whether or not the given problem can be modelled as a differential equation? Justify your answer.

Solution: Yes, the given problem can be modelled as a differential equation, since the given scenario involves rate of change of temperature and the change in temperature is continuous. The scenario is modelled using Newton's law of cooling as $\frac{dT}{dt} = -k(T - T_{env})$, where $\frac{dT}{dt}$ represents the rate of change of temperature, T is the temperature of the object at time t, T_{env} is the ambient temperature, k is the cooling or heating constant.

b. Identify the function embedded in the given problem, clearly indicating the dependent and independent variables.

Solution: T(t) is the function embedded in the given problem. Temperature (T) is the dependant variable and time (t) is the independent variable.

MATHEMATICAL MODELING

c. What is the typical count of arbitrary constants in the above problem? How do we determine these constants.

Solution: The above scenario can be modelled as a first order linear differential equation. The solution of first order differential equation has typical one arbitrary constant and it is determined by applying the initial condition in the general solution.

d. Is it possible to solve the above problem using numerical methods? If yes quote the methods.

Solution: Yes, Euler and modified Euler method, Runge-Kutta method.

e. Is it possible to solve the above problem using numerical methods, provided the initial condition is not specified? Justify your answer.

Solution: No, it is not possible to solve the above problem using numerical methods, provided the initial condition is not given. Because, the formula demands x_0 and y_0 which are from the initial conditions.

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PRACTICE PROBLEMS:

- Apply modified Euler's Method to the initial value problem
  $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ find $y(0.1)$ by taking h=0.05
  Ans : 1.1058

- Apply modified Euler's Method to find y(0.1) for the given initial value problem
  $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$
  Ans : 0.0005

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HOMEWORK PROBLEMS:

- Using Euler's and Modified Euler's method compute y(2.5) with h=0.25 from
  $\frac{dy}{dx} = \frac{x+y}{x}$, $y(2) = 2$
  Ans : 2.5808

- Given $y^1 - \sqrt{xy} = 2, y(1) = 1$, find y(1.25) using Modified Euler's method, take h=0.25 .
  Ans : 1.8132

- Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y(1) = 2$, find the approximate value of y at x=1.4 by taking step
  size h=0.2 by applying Modified Euler's method.
  Ans : 1.3619

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PRACTICE PROBLEMS:

- Use Runge- Kutta method to find y when x= 1.1 in steps of 0.1 given that

  $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 1.5$

  Ans : 1.8955

- Given $y' = x^2 - y$, $y(0) = 1$ find y(0.1) using Runge-Kutta method of Fourth order.

  Ans : 0.9052

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HOMEWORK PROBLEMS:

- Solve $(y^2 - x^2)\,dx = (y^2 + x^2)\,dy$ for x = 0.2 given that y=1 at x=0 initially, by applying R-K method of order four.

  Ans : 1.1960

- Use the R-K method with h=0.1 to obtain an approximation to y(1.5) for the solution of $y^1 = 2xy$, y(1) =1.

  Ans : 3.4544

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Module Outcomes:

At the end of the course, the student will be able to:

- Illustrate the knowledge of fundamental concepts of first order- first degree ordinary differential equations.
- Apply suitable techniques to solve given engineering and scientific problems related to first order- first degree ordinary differential equations based on the acquired knowledge.
- Analyze mathematical solutions of engineering and scientific problems related to first-order first-degree ordinary differential equations and predict their behavior in real-world scenario.

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THANK YOU