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NAAC A+ GRADE CGPA: 3.44

MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING

(A Unit of Rajalaxmi Education Trust ®, Mangalore)

Autonomous Institute affiliated to VTU, Belgavi, Approved by AICTE, New Delhi

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MODULE 5 : SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

- **Second-order linear differential equations:** Second-order linear differential equations with constant coefficients – Solution by Inverse differential operator - $e^{ax}, sinax/cosax, e^{ax}f(x)$. Application to Oscillations of a mass spring system and L-C-R circuit.

- **Numerical Solution of second order linear differential equations :** Runge-Kutta method and Milnes Predictor Corrector method to solve problems on oscillations of a mass spring system and L-C-R circuits.
- Solve second-order linear differential equations analytically by using the inbuilt function and analyze the solution graphically.

- Solve second-order linear differential equations by Runge - Kutta fourth order method.

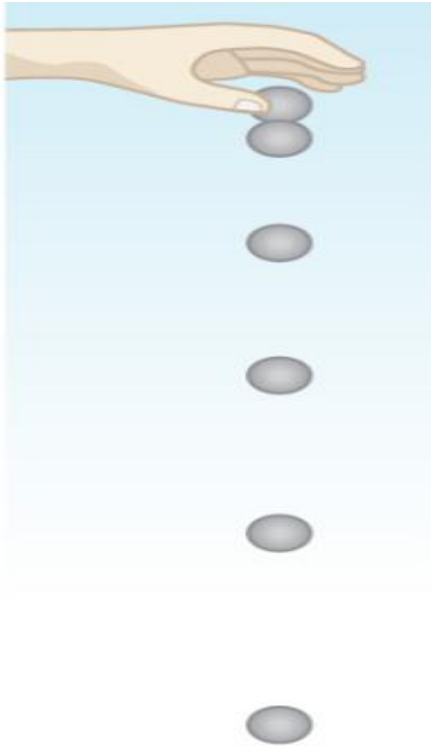
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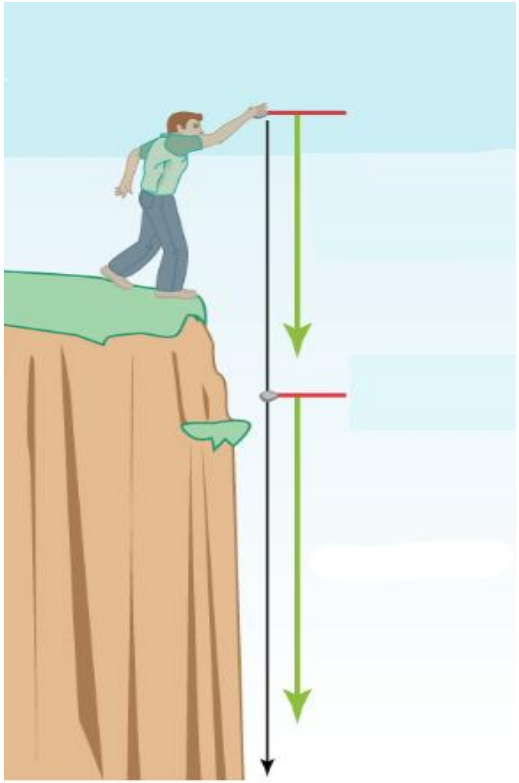
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Learning Objective

- Establish a foundation of modeling dynamical systems to obtain their solution by Analytical and Numerical methods.

FREE FALLING OBJECT





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How velocity relates to the falling object?

- Initially when we are about to drop the object, it starts with zero velocity.
- As the object falls under the influence of gravity, its velocity increases. This means it's speeding up as it moves downward.
- The velocity is positive (indicating downward motion) and increases as long as the object is falling freely without any other forces acting on it, like air resistance.
- If you were to measure the velocity of the object at different moments during its fall, you would find that it's getting faster and faster in the downward direction.

How Velocity Changes With time?

This can be represented in terms of ODE as

$$v(t) = \frac{dy}{dt}, \qquad y \rightarrow position$$

$$\frac{dv(t)}{dt} = g$$

$$\Rightarrow \frac{d\left(\frac{dy(t)}{dt}\right)}{dt} = g$$

$$\frac{d^2y(t)}{dt^2} = g \text{ (external force will be in RHS)}$$

is called Second-order linear differential equations as the order of differentiation is two.

Here the change of velocity depends on change of position (first order differentiation) with respect to time (independent variable).

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How is second order linear differential equation connected to falling object ?

- Initially, when you drop the object, it has an initial velocity of 0 (it's not moving). So, at the beginning, dv/dt is zero.

- As time passes, the object starts to accelerate due to gravity, and this acceleration causes its velocity to increase positively.

- The object's velocity increases more and more over time due to the constant acceleration, until it reaches a point where it stops falling and begins to move upward (if there's something to stop it, like the ground or your hand).

Differential Equations - Quiz

1. The general form of a Second-order linear differential equation is:

A) $y'' + p(x)y' + q(x)y = \varphi(x)$
B) $y' + p(x)y = q(x)$
C) $y'' = p(x)y$
D) $y'' + y = 0$

Ans – A)

2. If $\varphi(x) = 0$, the differential equation is called:

A) Non-linear
B) Non-homogeneous
C) Homogeneous
D) Linear with constant coefficients

Ans – C)

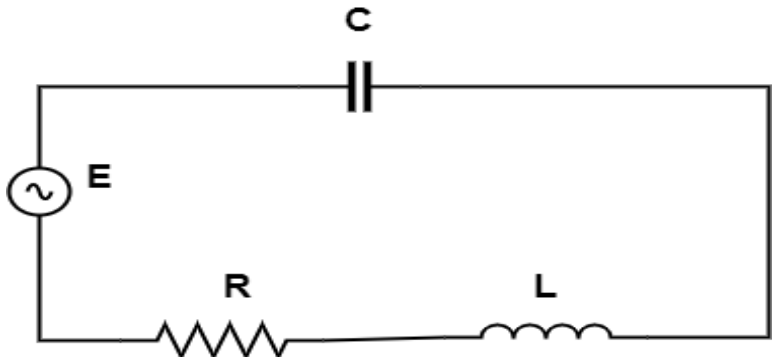
3. The term $\varphi(x)$ in the equation $y'' + p(x)y' + q(x)y = \varphi(x)$ represents:

A) Internal resistance
B) External force or input
C) Frictional term
D) Mass term

Ans – B)

LCR Circuit

Consider an Electrical circuit which consists of Inductance(L), Resistance (R), Capacitance (C). Let I be the current flowing through this circuit at any time t.



By Kirchoffs' law,

$$\boldsymbol{E(t)} = L\frac{dI}{dt} + RI + \frac{q}{C}$$

By rearranging,

$$\boxed{\frac{\boldsymbol{d^2q}}{\boldsymbol{dt^2}} + \frac{\boldsymbol{R}}{\boldsymbol{L}}\frac{\boldsymbol{dq}}{\boldsymbol{dt}} + \frac{\boldsymbol{q}}{\boldsymbol{LC}} = \frac{\boldsymbol{E(t)}}{\boldsymbol{L}}} \quad \text{-------(1)}$$

LCR Circuit

The circuit is said to be free if E=0.

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \qquad \text{--------(2)}$$

Equations (1) and (2) are called second order ODE with constant coefficients.

General Form of Second Order Linear Differential Equation

Linear second order differential equation is in the form of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = \phi(x) \quad \text{--------(2a)}$$

OR

$$y'' + p(x)y' + q(x)y = \phi(x)$$

If cannot be written in the above form, it is non-linear.

- If $\phi(x) = 0$ in eqn (2a), the equation is called homogenous.
- If $\phi(x)$ is non-zero in eqn (2a), it is non-homogenous.
- Eqn (2a) LDE with variable coefficients

Linear Second Order DE With Constant Coefficient.

The general form is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \phi(x) \quad \text{----------(3)}$$ where a, b, c are constants

the non-homogeneous term $\phi(x)$ often represents external forces, inputs or disturbances that affect the system

Homogenous form of (3) can be written as

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{----------} (4)$$

In an LCR circuit, the solution describes the behavior of the circuit with respect to changes in voltage or current.

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Solution of Second Order ODE

General solution = Complementary solution + particular integral

$$y = y_c + y_p$$

The above terms represent

- **General solution (y)** : Family of solutions that includes all possible solutions to the differential equation (as it involves arbitrary constants)
- The general solution typically includes both the steady-state behavior (**Complementary solution)** and transient behavior (**Particular integral)** of the dynamical system.

Solution of Second Order ODE

- **Complementary solution (y_c)** : Solution to the homogeneous differential equation.
- The complementary solution also known as the natural response or transient response of a dynamical system & and is often derived from the characteristic equation of the differential operator.
- It describes the behavior of the system when it is subjected to initial conditions (e.g., initial voltages or currents) and then left to evolve on its own without any external sources (e.g., voltage or current sources).
- **Particular integral or Particular Solution (y_p)** : Solution to the non-homogeneous part of the differential equation.
- The particular integral represents the forced response of the dynamical system which is caused by external sources (Ex. voltage or current sources). It describes how the circuit responds to external inputs.
- It accounts for the non-zero forcing function (the non-homogeneous part). The form of $\boldsymbol{y_p}$ depends on the specific non-homogeneous term, and it represents how the external input affects the system.

Complementary Solution

On solving the homogeneous D.E $\boldsymbol{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$ -------(1)

replace $\frac{d}{dx}$ by D, $\frac{d^2}{dx^2}$ by D^2 etc, where D is called the differential operator in (1)

$(1) \Rightarrow (aD^2 + bD + c)y = 0$ (Alternate form of (1))

This can also be written as $f(D)y = 0$

Solve the auxiliary equation $f(m) = 0$

which is obtained by replacing D by m in $f(D) = 0$

Let m_1 and m_2 be the roots of $f(m) = 0$

- m_1 and m_2 may be real distinct
- m_1 and m_2 may be real repeated
- m may be complex root (appear as complex conjugates)

Complementary Solution

Depending on the nature of roots (m_1) and (m_2), following are the cases

- Real and distinct roots

$$\mathbf{y = y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}, (m_1 \neq m_2)}$$

- Real and repeated roots (m)

$$\boldsymbol{y = y_c = (c_1 + c_2 x) e^{mx}}$$

- Complex roots $(\alpha \pm i\beta)$

$$\boldsymbol{y = y_c = e^{\alpha x}[c_1 cos(\beta x) + c_2 sin(\beta x)]}$$

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Quiz

1.The general solution of a second-order linear ODE is:

A) Particular solution

B) Complementary solution

C) Complementary + Particular solution

D) Homogeneous solution only

Ans:- C)

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Quiz

2. The particular integral represents:

A) Transient response
B) Steady-state (forced) response
C) Random fluctuation
D) Homogeneous behavior

Ans:- B)

3. The complementary function represents:

A) The forced response
B) The natural response
C) The random motion
D) External input behavior

Ans:- B)

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4. If the auxiliary equation has repeated roots, the complementary function is:

A) $C^1 e^{\{m_1 x\}} + C^2 e^{\{m_2 x\}}$

B) $(C^1 + C^2 x) e^{\{mx\}}$

C) $C_1 \cos bx + C_2 \sin bx$

D) $C^1 e^{\{ax\}}$

Ans:- B)

5. If roots are complex $\alpha \pm i\beta$, the complementary function is:

A) $e^{\{\alpha x\}}(C_1 \cos \beta x + C_2 \sin \beta x)$

B) $C^1 e^{\{\beta x\}} + C^2 e^{\{\alpha x\}}$

C) $C_1 \cos \alpha x + C_2 \sin \alpha x$

D) $e^{\{i\beta x\}}$

Ans:- A)

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Inverse differential operator and particular integral (particular solution)

- If D is the differential operator, then the inverse differential operator, denoted by D^{-1} or $\frac{1}{D}$ is an operator such that $DD^{-1}\big(f(x)\big) = f(x)$.

- The terms $\frac{1}{D}, \frac{1}{D^2}$ stand for successive integration.

- Inverse differential operators are used to obtain the particular integral of a non-homogenous second order ODE.

Particular Integral (Particular Solution)

Particular Integral $y_p = \frac{\emptyset(x)}{f(D)}$, where $\emptyset$ (x) is a function of x.

This is obtained based on the following cases of $\emptyset$ (x):

Case 1: If $\emptyset(x) = e^{ax}$, then $y_p = \frac{e^{ax}}{f(D)}$

Replace D *by* a(co-efficient of x in exponential term)

$$\Rightarrow y_p = \frac{e^{ax}}{f(a)}, where\ f(a) \neq 0$$

When $f(a) = 0$:

Differentiate $f(D)$ w.r.t D & multiply numerator by x $(Independent\ var)$

$$\text{then } y_p = \frac{e^{ax}x}{f^1(a)},\ replace\ D\ by\ a, \text{where } f^1(a) \neq 0$$

If $f'(a) = 0$, then $y_p = \frac{e^{ax}x^2}{f''(a)}$ where $f''(a) \neq 0$ and so on, till we get a constant value in denominator

Case 2: If $\emptyset(x) = cosax$ (or) $sinax$

then $y_p = \frac{\cos ax / \sin ax}{f(D)}$

To solve the above replace $D^2 = -a^2$ and solve the same for the denominator not equal to zero and if the denominator becomes zero,

we have $y_p = \frac{cosax}{D^2+a^2} = \frac{xcosax}{2D}$ and $y_p = \frac{sinax}{D^2+a^2} = \frac{xsinax}{2D}$

(After Replacing D^2 by $-a^2$, $f(D)$ transforms into the form $\alpha D \pm \beta$ (α and β are constants) in which case we multiply both the numerator and denominator by $\alpha D \mp \beta$ for proceeding further in obtaining the PI.)

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Case 3: If $\emptyset(x) = e^{ax} f(x)$, then $y_p = \frac{e^{ax} f(x)}{f(D+a)}$.

If e^{ax} is multiplied with some function of x, we need to first shift D to D+a.

Then compute PI follow Case 2

WORKED EXAMPLES

1) $y'' + 4y' + 4y = 0$ given that $y = 0, y' = -1$ at $x = 1$.

Ans: $D^2y + 4Dy + 4y = 0$

$$[D^2 + 4D + 4]y = 0$$

$$f(D)y = 0$$

$A.E. \rightarrow m^2 + 4m + 4 = 0$

$m = -2, 2\ (repeated)$

$G.S. \Rightarrow y_c = [c_1 + xc_2]e^{-2x}$ ---------- (1)

Since $RHS = 0, y_p = 0$

Now, $y = [c_1 + xc_2]e^{-2x}$

$$y' = [c_1 + xc_2]e^{-2x}.(-2) + (0 + c_2)\, e^{-2x}$$

$$y' = [c_1 + xc_2](-2e^{-2x} + c_2e^{-2x}) \quad \text{----------(2)}$$

Put $y = 0\ at\ x = 1$

$0 = (c_1 + c_2)$

$c_1 = -c_2$

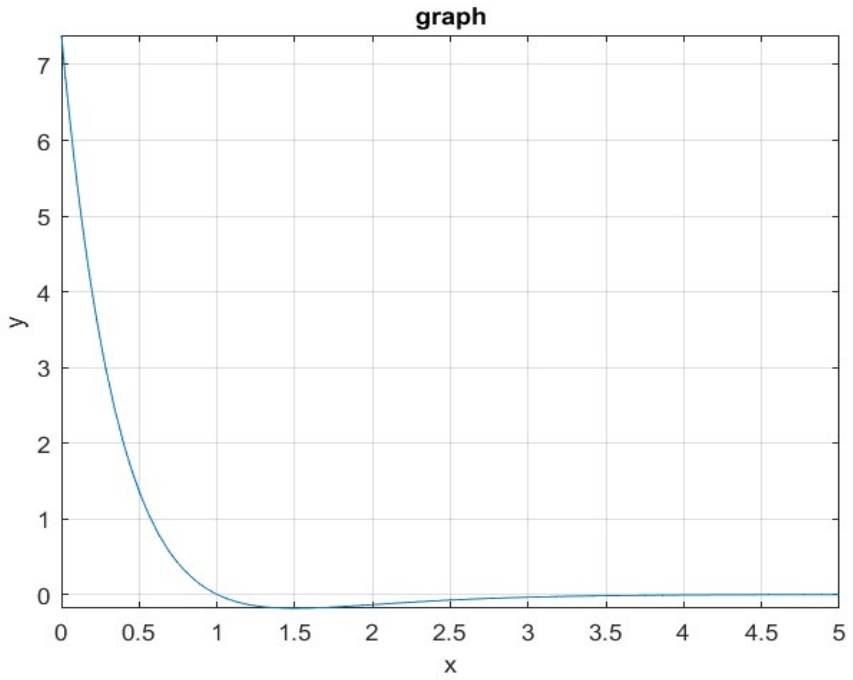
Now, put $y' = -1$ & $x = 1$

from (2),

$-1 = (c_1 + c_2)(-2e^{-2}) + c_2 e^{-2}$

Since, $c_1 = -c_2$

$c_2 = -e^2$



WORKED EXAMPLES

2) Determine the complementary solution of the differential equation: $y'' - 3y' - 4y = 0$

Sol : Aux-Eq : $m^2 - 3m - 4 = 0 \Longrightarrow m = 4, -1$

$$\therefore y_c = C_1 e^{4x} + C_2 e^{-x}$$

3) Identify the roots corresponding to the differential equation: $y'' + (a+b)y' + aby = 0$ and hence write its complementary solution.

Sol : Aux-Eq :

$$m^2 + (a+b)m + ab = 0 \Longrightarrow (m+a)(m+b) = 0$$

$$\Longrightarrow m = -a, -b$$

$$\therefore y_c = C_1 e^{-ax} + C_2 e^{-bx}$$

WORKED EXAMPLES

4) Solve $(D^2 + 4D + 3)y = e^{-x}$

Ans: $y = c_1 e^{-x} + c_2 e^{-3x} + \frac{xe^{-x}}{2}$

5) $(D-2)^2 y = 8(e^{2x} + \sin 2x)$

Ans: $y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x$

6)Solve $(D^2 - 4D + 13)y = cos2x$

Ans : $y = e^{2x}(c_1 cos3x + c_2 sin3x) + \frac{9cos2x - 8sin2x}{145}$

WORKED EXAMPLES

7) $(D^2 - 2D + 5)y = e^{2x} sinx$

Ans: $y = e^x(c_1 cos2x + c_2 sin2x) + \frac{e^{2x}}{10}(2sinx - cosx)$

8) $(D^2 - 3D + 2)y = 2e^x \cos\frac{x}{2}$

Ans: $y = c_1 e^x + c_2 e^{2x} - \frac{8}{5}e^x\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)$

9) Solve $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

Ans :$y = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7e^{-3x}}{25} - \frac{log\ 2}{9}$

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PRACTICE PROBLEMS

1) $(6D^2 + 17D + 12)y = e^{-x}$

Ans: y=$C_1 e^{-\frac{3x}{2}} + C_2 e^{-\frac{4x}{3}} + e^{-x}$

2) $(D^2 + 2D + 1)y = 2e^{3x}$

Ans: $y = (C_1 + C_2 x)e^{-x} + \frac{e^{3x}}{8}$

3) $(D^2 - 5D + 6)y = e^{4x}$

Ans: $y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2}e^{4x}$

4) $(D^2+4)y = sin3x + \cos 2x$

Ans: y=$C_1 cos2x + C_2 sin2x - \frac{1}{5} sin3x + \frac{x}{4} sin2x$

5) $(D^2+2D + 5)y = 6\ sin2x + 7\ cos2x$

Ans:$e^{-x}(C_1 sin2x + C_2 cos2x) + 2sin2x\ - cos2x$

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PRACTICE PROBLEMS

6) Given the complementary function below. Identify the nature of the roots and write the roots.

i) $y_c = (C_1 + C_2x)e^{-2x}$

(ANS: Real and repeated roots, $m = -2, -2$)

ii) $y_c = (C_1 cos5x + C_2 sin5x)e^{-x}$

(ANS: Complex roots, $m = -1 \pm 5i$)

iii) $y_c = (C_1e^{-9x} + C_2e^{x})$

(ANS: Real and distinct roots, $m = -9, 1$)

iv) $y_c = (C_1 + C_2x)e^{7x}$

(ANS: Real and repeated roots, $m = 3,3$)

PRACTICE PROBLEMS

7) Given the general solution $y = C_1 cos2x + C_2 sin2x - \frac{1}{5} sin3x + \frac{x}{4} sin2x$

i) Identify the complementary function (C.F.) of the given solution.

(ANS: $y_c = C_1 cos2x + C_2 sin2x$)

ii) Identify the particular integral (P.I.) of the given solution

(ANS: $y_p = -\frac{1}{5} sin3x + \frac{x}{4} sin2x$)

iii) State the number of arbitrary constants present in the given solution.

(ANS: 2)

iv)Identify the form of the roots of the auxiliary equation corresponding to the complementary function.

(ANS : Complex roots, $m = \pm 2i$)

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PRACTICE PROBLEMS

8) Solve the differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

ANS : $x = (C_1 + C_2 t)e^{-3t}$

9) Solve the differential equation : $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, Given that initial conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$.

ANS : $x = C_1 e^t + C_2 e^{2t}$, applying Initial condition, $x = 0$

10) Solve the differential equation: $(4D^2 + 4D - 3)y = e^{2x}$

ANS : $y = C_1 e^{\frac{x}{2}} + C_2 e^{-\frac{3x}{2}} + \frac{e^{2x}}{21}$

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PRACTICE PROBLEMS

12) Solve the differential equation: $(\mathrm{D}-2)^2 y = 8e^{2x}$

ANS : $y = (C_1 + C_2 x)e^{2x} + 4x^2 e^{2x}$

13) Solve the differential equation: $(D^2 + 3D + 2)y = 5$

ANS : $y = C_1 e^{-x} + C_2 e^{-2x} + \frac{5}{2}$

14) Solve the differential equation: $y'' + 4y' + 4y = 0$. given that $y = 0, y' = -1$ at $x = 1$

ANS : $y = (e^2 - e^2 x)e^{-2x}$

15) Solve the differential equation: $(D-2)^2 y = 8(e^{2x} + \sin 2x)$.

ANS: $y = (C_1 + C_2 x + 4x^2)e^{2x} + cos2x$

PRACTICE PROBLEMS

16) $(D^2 + 4)y = sinx + sin2x$

Ans: $C_1 sin2x + C_2 cos2x + \frac{sinx}{3} - \frac{xcos2x}{4}$

17) $(D^2 + 16)y = e^{-3x} + cos4x$

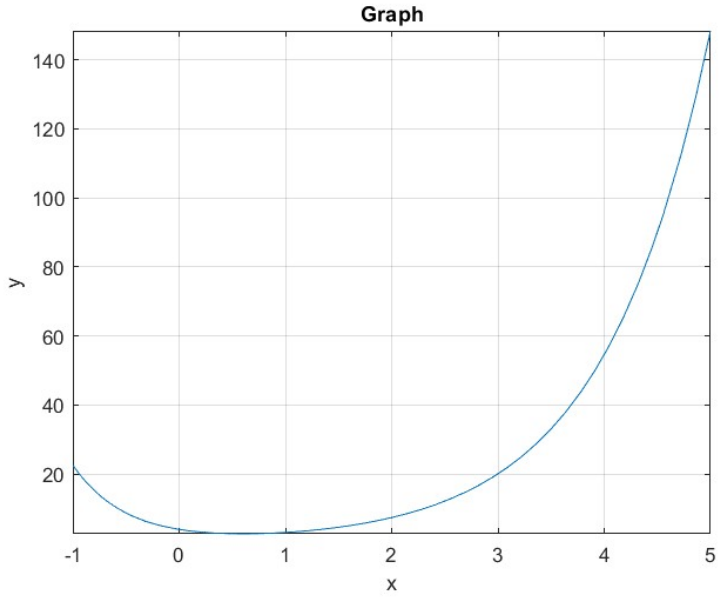
Ans: $C_1 cos4x + C_2 sin4x + \frac{1}{25} e^{-3x} + \frac{x}{8} sin4x$

18) $(D^2 - 4D + 3)y = e^x cos2x + cos3x$

Ans: $C_1 e^x + C_2 e^{3x} - \frac{e^x}{8}[sin2x + cos2x] - \frac{1}{30}[2sin3x + cos3x]$.

HOMEWORK PROBLEMS

1) $y''+y'-2y=0$; Given $y(0)=4; y'(0)=-5$ $\quad$ $Ans:\ y=e^{x}+3e^{-2x}$



2) $\frac{d^2y}{dx^2}-4dydx+4y=e^{2x}+cos(2x)+4$

Ans: $y=(c_1+c_2x)e^{2x}+\frac{x^2e^{2x}}{2}-\frac{sin(2x)}{8}+1$

HOMEWORK PROBLEMS

3)Interpret the general solution of the differential equation: $y'' - 4y' + y = 0$.

ANS: $y = C_1 e^{(2+\sqrt{3})x} + C_2\ e^{(2-\sqrt{3})x}$

4) Evaluate the particular integral of $(4D^2 + 4D - 3)y = e^{2x}$

ANS: $y = \frac{1}{21} e^{2x}$

5) Solve the differential equation $(D^2 + 4)y = \cos 2\,x$

ANS : $y = C_1 cos2x + C_2 sin2x + \frac{xsin2x}{4}$

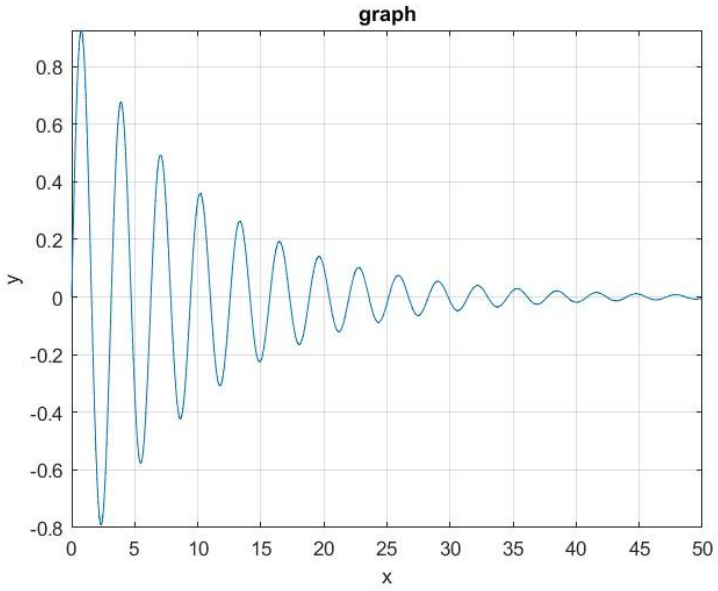
HOMEWORK PROBLEMS

6) $y'' - 3y' + 2y = 2sinx\ cosx$

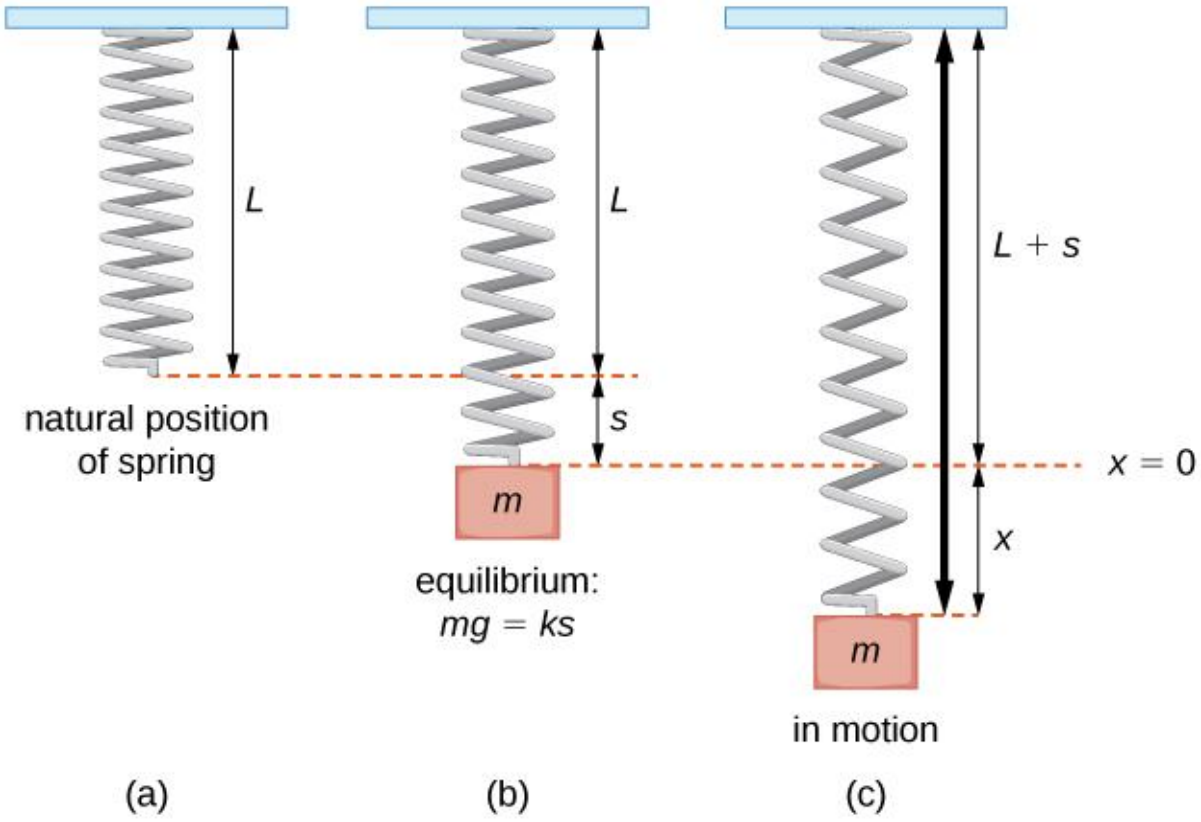
Ans: $y = c_1e^x + c_2e^{2x} + \frac{(3cos2x - sin2x)}{20}$

7) Solve $y'' - 0.2y' + 4.01y = 0$; Given $y(0) = 0; y'(0) = 2$

$Ans:\ y = e^{-0.1x}sin2x$



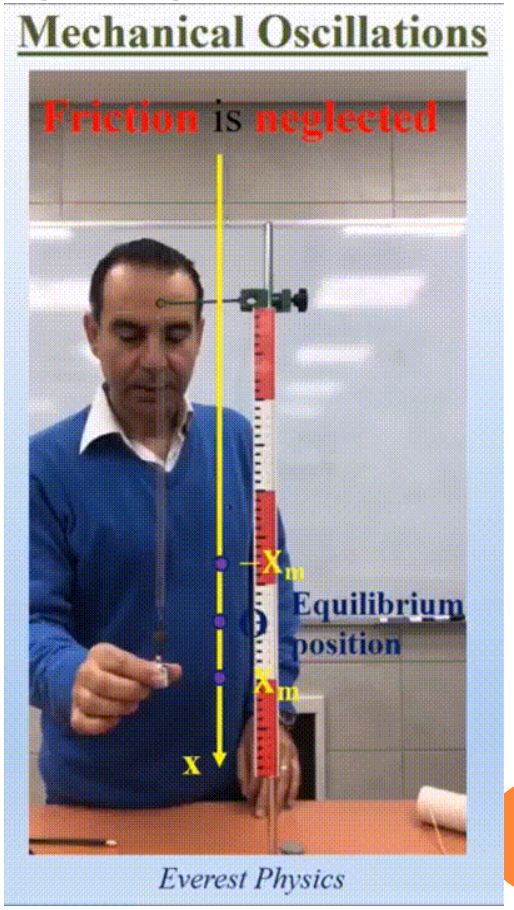
Mass Spring Problem



Different Cases of Oscillating system

- **Free Undamped Oscillation**: This is like giving something a push to the object or system and letting it swing back and forth on its own, without slowing it down.

It just keeps moving forever at the same size because there's no resistance to make it stop.



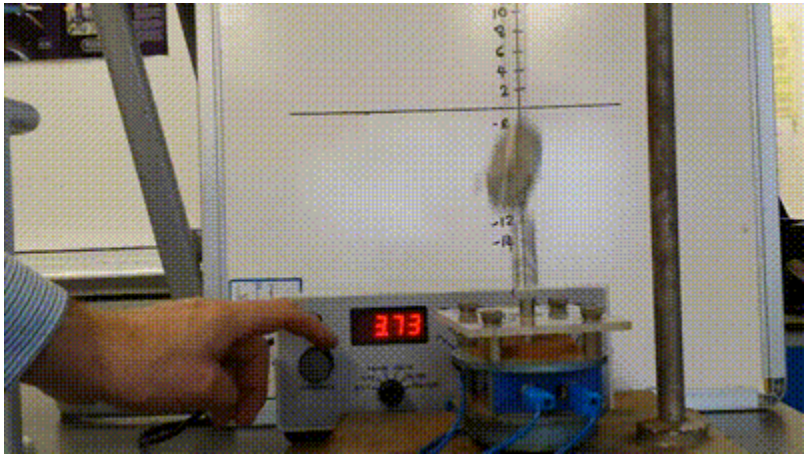
Different Cases of Oscillating system

- **Free Damped Oscillation:** Imagine you push an object or system, but there's a bit of resistance, like friction or air slowing it down. Over time, it loses energy and eventually stops moving completely. It's that slowing-down effect we see in real life when things don't move forever.



Different Cases of Oscilatory system

- **Forced Undamped Oscillation:** Here, you keep giving steady pushes an object or system to keep moving back and forth, and there's no resistance to hold it back. If you push in sync with its natural rhythm, it'll go bigger and bigger each time, because nothing is working against it.



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Different Cases of Oscilatory system

- **Forced Damped Oscillation:** This is when you keep pushing an object or system to keep it moving, but there's also some resistance (like friction) slowing it down. It reaches a balance where the movement stays steady your pushes match up with the resistance, so it doesn't get bigger or smaller, just stays constant.

Oscillatory systems

	Mass Spring	LCR
Free Undamped	$m.x'' + kx = 0$	$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$
Free Damped	$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$	$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$
Forced Undamped	$m.x'' + kx = f(t)$	$L\frac{d^2q}{dt^2} + \frac{q}{C} = E(t)$
Forced Damped	$x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{f(t)}{m}$	$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E(t)}{L}$

Where m=Mass of the object ; k=Spring constant; c=Damping constant;

$f(t)$ =External force. $x'' = \frac{d^2x}{dt^2}$; $x' = \frac{dx}{dt}$

L=Inductance; R=Resistance; C=capacitance, E=emf.

- Formulate the second order linear differential equation for a mass-spring system using the following given values and write the type of Oscillation for a mass-spring system or $L - C - R$ with the following given values:

i) $m = 2kg$, $k = 18N/m$ and no external force.

ANS: $2\frac{d^2x}{dt^2} + 18x = 0$

Free Undamped Oscillation

ii) $m = 4kg$, $c = 8Ns/m$, $k = 36N/m$ and $f(t) = 0$.

ANS : $4\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 36x = 0$

Free Damped Oscillation

iii) $L = 2H, C = 0.5F, R = 0$ and $E(t) = 0$

ANS : $q'' + \frac{1}{2}q' + q = 0$

Free Damped $L - C - R$ circuit

iv) $m = 5kg, k = 45N/m, c = 0$ and the external force is $f(t) = 20cos3t.$

ANS: $x'' + \frac{7}{2}x' + \frac{11}{2}x = \frac{8e^{-t}}{2}$

Forced Damped Oscillation

v) $m = 5kg, k = 45N/m, c = 0$ and the external force is $f(t) = 20cos3t.$

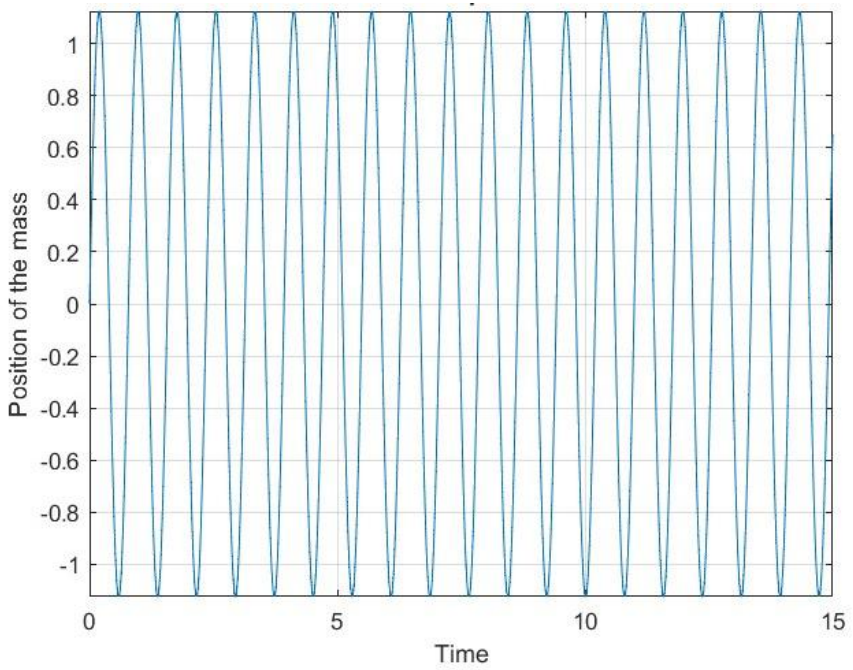
ANS : $\frac{d^2x}{dt^2} + 9x = 4cos3t$

Forced Undamped Oscillation

WORKED PROBLEMS

1) A spring with a mass of 2 kg and spring constant of 128 N/m is stretched and then released from rest with an initial velocity of 0 m/s. given that the initial displacement is $x(0) = 0.2m$. Determine the position of the mass at any time t, where $x(t)$ represents displacement from the spring's natural length.

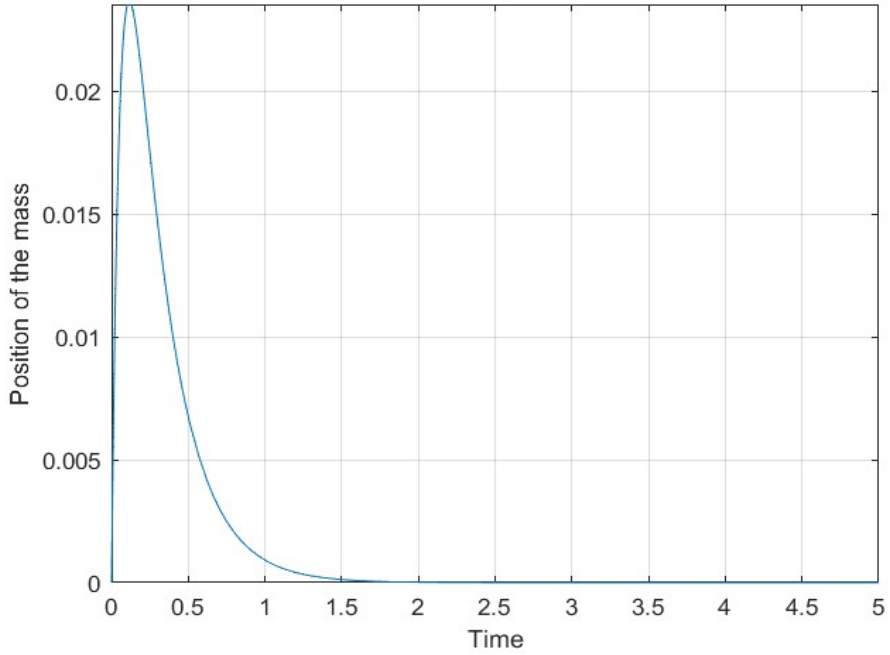
(Ans: $x(t) = 0.2\ cos8t$)



This graph represents an **undamped system**.

2) A spring with a mass of 2 kg and spring constant of 128 N/m is stretched and then released. It is immersed in a fluid with damping constant c=40kg/s. The mass starts from equilibrium position and is given an initial velocity of 0.6m/s. Determine the position of the mass at any time t, where $x(t)$ represents the displacement from the spring's natural length.

(Ans: $x(t) = 0.05(e^{-4t} - e^{-16t})$)



This is Free Damped Oscillations action.

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

On substituting the given data , $x'' + \frac{40}{2}x' + \frac{128}{2}x = 0$

$$x'' + 20x' + 64x = 0$$

$$F(D)x = 0$$

Auxiliary Equation -

$$m^2 + 20m + 64 = 0$$

$$m_1 = -4, m_2 = -16$$

The roots are real & distinct

$$x_c = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

$$x_c = C_1 e^{-4t} + C_2 e^{-16t} \text{--------(1)}$$

Initial Condition a) $at\ t = 0\ , x(0) = 0$

$$(1) \text{------>} 0 = C_1 + C_2 \text{---------(2)}$$

Differentiate eq (1) and apply Initial Condition b)

$$at\ t = 0\ , x'(0) = 0.6$$

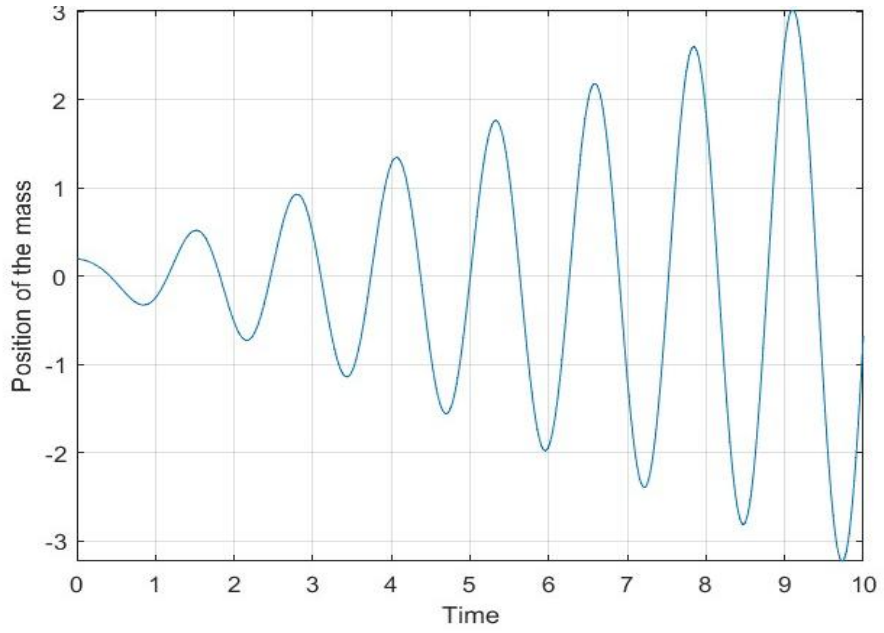
$$(1) \text{------>} 0.6 = C_1\,(-4)e^{-4(0)} + C_2(-16)e^{-16(0)}$$

$$-4C_1 - 16C_2 = 0.6 \text{------} -(3)$$

On solving (2) and (3) we get $C_1 = 0.05\ , C_2 = -0.05$

3) A $\boldsymbol{3kg}$ object is attached to a spring with spring constant 75 N/m. A forcing function $F(t) = 10\cos(5t)$ is applied to the object. The object is initially displaced 0.2m downward from its equilibrium position and given an initial velocity of 0.1 m/s upward. Find the displacement at any time t.

(Ans: $x(t) = 0.2\cos 5t - 0.02\sin 5t + \frac{1}{3}(tsin5t)$)



4) Determine the charge on the capacitor at any time t > 0 in circuit series having an emf E(t) = 100 sin60t V, a resistor of 2Ω, and inductor of 0.1H and capacitor of $\frac{1}{260}$F, if the initial current and charge on the capacitor are both zero.

(Ans: $q(t) = e^{-10t}(0.4918\cos 50\,t + 0.5901\,sin50t) - (\frac{30}{61}cos60t + \frac{25}{61}sin60t)$)

5) An electric circuit consists of an inductance of 0.1H a resistance of 20Ω and a condenser of 25 μF. Find the charge q at any time t, given that q(0) = 0.05 and i(0)=0.

(Ans: $q(t) = e^{-100t}(0.05\cos(624.5\,t) + 0.008\sin(624.5t))$)

HOMEWORK PROBLEMS

1)A 4$\boldsymbol{kg}$ object is attached to a spring with spring constant 100 N/m. The system experiences a damping with a damping constant 60 N/m. A forcing function $F(t) = 12\cos(4t)$ is applied to the object. The object is initially displaced 0.15m downward from its equilibrium position and given an initial velocity of 0.2 m/s upward. Find the displacement at any time t.

Ans – $0.137e^{-1.9098} + 0.011e^{-13.09} + 0.007cos4t + 0.048sin4t$

2) A **3$\boldsymbol{kg}$** object is attached to a spring with spring constant 75 N/m. The system experiences a damping with a damping constant c=90 N/m. A forcing function $F(t) = 10\cos(5t)$ is applied to the object. The object is initially displaced 0.2m downward from its equilibrium position and given an initial velocity of 0.1 m/s upward. Find the displacement at any time t.

3) Find the charge on the capacitor in an RLC series circuit where L=5/3 H, R=10Ω, C=1/30 F, and E(t)=300 V. Assume the initial charge on the capacitor is 0 C and the initial current is 9 A.

(Ans: q(t)= $e^{-3t}(-10\cos(3\,t) - 7\sin(3\,t)) + 10)$

Numerical Solution of Second Order ODE

Most real-world ordinary differential equations cannot be solved exactly and are often analyzed qualitatively, but this may not yield accurate results. To obtain precise approximate solutions for second-order ODEs, numerical methods such as the Runge-Kutta method and Milne's Predictor-Corrector method are commonly employed.

Consider a Second order ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = \varphi(x) \text{ ---- } -(1)$$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

then, (1) => $\frac{dz}{dx} = g(x, y, z)$ to the initial conditions

$$y(x_0) = y_0;\ y'(x_0) = y_0' = z(x_0) = z_0$$

Runge-Kutta Method (R-K Method):

The Runge-Kutta method involves iteratively updating the solution from an initial point to approximate the solution at successive points in the domain. It is based on weighted averages of function evaluations at multiple intermediate points within each step. We can find the solution at $x = x_1$ by following,

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4] \quad \text{where}$$

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \qquad l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \qquad l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

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Milne's Predictor-Corrector Method:

It is a predictor-corrector method, which means that it uses two steps to approximate the solution. In the predictor step, a linear approximation to the solution is made. This approximation is then used to compute the solution at the next time step. In the corrector step, the actual solution is computed at the next time step and compared to the predicted solution. We can find the solution at $x = x_1$ by following,

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3); \qquad z_4^p = z_0 + \frac{4h}{3}(2g_1 - g_2 + 2g_3)$$

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right); \qquad z_4^c = z_2 + \frac{h}{3}\left(g_2 + 4g_3 + g_4^p\right)$$

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Problems:

1) Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$ for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 1;\ y'(0) = 0$ using R-K Method of 4th Order.

$Ans: y(0.1) = 1.0053$

2) Solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 3\ ;\ \ y'(0) = 0$ using R-K Method of 4th Order.

$Ans: y(0.1) = 2.9849$

3) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ for $x = 0.1$ correct to 4 decimal places using the conditions $y(0) = 2\ ;\ \ y'(0) = 1$ using R-K Method of 4th Order.

$Ans: y(0.1) = 2.0716$

4) Apply the Milne's predictor-corrector method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$

X	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

Let $\frac{dy}{dx} = z = f(x, y, z)$,

With this substitution, $\frac{dz}{dx} = 1 - 2yz = g(x, y, z)$

x	y	z	$f(x, y, z)$	$g(x, y, z)$
0	0	0		
0.2	0.002	0.1996	$f_1 = 0.1996$	$g_1 = 0.1996$
0.4	0.0795	0.3962	$f_2 = 0.3962$	$g_2 = 0.3962$
0.6	0.1762	0.5689	$f_3 = 0.5689$	$g_3 = 0.5689$

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^p = 0 + \frac{4(0.2)}{3}(2(0.1996) - 0.3962 + 2(0.5689)) = 0.3039$$

$$z_4^p = z_0 + \frac{4h}{3}(2g_1 - g_2 + 2g_3)$$

$$z_4^p = 0 + \frac{4(0.2)}{3}(2(0.992) - 0.9370 + 2(0.7995)) = 0.7056$$

$f_4^p = f(x_4, y_4, z_4) = z_4 = 0.7056$, $g_4^p = g(x_4, y_4, z_4) = 1 - 2y_4z_4$

$$g_4^p = 1\text{-}2(0.3039)(0.7056) = 0.5711$$

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p); \qquad z_4^c = z_2 + \frac{h}{3}(g_2 + 4g_3 + g_4^p)$$

$y_4^c = 0.3047$ $z_4^c = 0.7099$

4) Apply the Milne's predictor-corrector method to compute $y(0.8)$ given that $y'' = 1 - 2yy'$

X	0	0.2	0.4	0.6
Y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

Ans: $y_4^{(p)} = 0.3039; z_4^{(p)} = 0.7050; y_4^{(c)} = 0.3047$

5) Apply the Milne's predictor-corrector method to compute $y(0.4)$ given that $y'' = 1 + y'$

X	0	0.1	0.2	0.3
Y	1	1.1103	1.2427	1.3990
$y' = z$	1	1.2103	1.4427	1.6990

Ans: $y_4^{(p)} = 1.5835; z_4^{(p)} = 1.9835; y_4^{(c)} = 1.5835$

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Practice Problems

1) Apply the Milne's predictor-corrector method to compute $y(0.8)$ given that $y'' = 2yy'$

X	0	0.2	0.4	0.6
Y	0	0.2027	0.4228	0.6841
$y' = z$	1	1.041	1.179	1.468

$Ans : y_4^{(p)} = 1.0237; z_4^{(p)} = 2.0304; y_4^{(c)} = 1.02823$

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Homework problems

1) Solve $\frac{d^2y}{dx^2} - x\frac{dy}{dx} = y$ for $x = 0.2$ correct to 4 decimal places using the conditions $y(0) = 1$; $y'(0) = 0$ using R-K Method of 4th Order.

Ans: $y(0.2) = 0.9802; y' = -0.196$

2) Use R-K Method solve $y'' - xy + 4y = 0; y(0) = 3; y'(0) = 0$ at $x = 0.1$

Ans:2.9399

3) Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 0$ given that $y(0) = 1; y'(0) = 0$; for y(0.1) using R-K Method.

Ans: 1.0053

4) Solve $y' = x + z,\ z' = x - y$ for $x = 0.1$ given that $y = 0, z = 1\ at\ x = 0$ using R-K Method

Ans:0.1050

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3) Apply the Milne's predictor-corrector method to compute $y(0.4)$ given that $y'' + xy' + y = 0$

X	0	0.1	0.2	0.3
Y	0	0.995	0.9802	0.956
$y' = z$	0	-0.0995	-0.196	-0.2863

Ans: $y_4^{(p)} = 0.9231; z_4^{(p)} = 0.9232; y_4^{(c)} = -0.3692$

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THANK YOU