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NAAC A+ GRADE CGPA: 3.44

MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING

(A Unit of Rajalaxmi Education Trust ®, Mangalore)

Autonomous Institute affiliated to VTU, Belgavi, Approved by AICTE, New Delhi

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MODULE -2: VECTOR SPACE

Definition and examples, subspace, Linearly independent and dependent sets, linear span, Basis and dimension, Inner product spaces, Projection and orthogonality.

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LEARNING OBJECTIVES:

- Understand the formal definition of a vector space and its key properties.
- Apply linear span to solve problems related to spanning sets, dependence, and independence of vectors.
- Study Eigen values and Eigenvectors of square matrices, along with changing coordinates from one basis to another.

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FIELD:

A **Field F** is a non-empty set of elements equipped with two operations: addition (+) and multiplication (·) satisfying the following properties for all elements a, b and c in the field:

1. **Closure:** For any $a, b \in F$, $a + b \in F$ and $a \cdot b \in F$
2. **Commutativity:** Addition and multiplication are commutative, i.e., $a + b = b + a$ and $a \cdot b = b \cdot a$ for all a and b in the field.
3. **Associativity:** Addition and multiplication are associative, i.e., $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

4. Additive Identity:

There exists an element $0 \in F$ such that $a + 0 = a$

5. Multiplicative Identity: There exists an element $1 \in F$ such that $a \cdot 1 = a$

6. Additive Inverses: For every $a \in F$, there exists an element $-a \in F$ such that $a + (-a) = 0$.

7. Multiplicative Inverses: For every $a \neq 0 \in F$, there exists an element $a^{-1}\ (or\ \frac{1}{a}) \in F$ such that $a \cdot a^{-1} = 1$.

Common examples of fields include:

- the set of real numbers ($\mathbb{R}$)
- the set of complex numbers (C)
- the set of rational numbers (Q).

These sets satisfy all the properties mentioned above, making them fields.

Set of all real numbers ($\mathbb{R}$) is the most commonly used field.

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THINK!

Whether the **Set of Integers (I)** forms a field?

- **No, it is not a Field. Because of Non-Closure under Multiplicative Inverses.**
- **Closure under multiplicative inverses means for any non-zero element $a \in I$ its inverse $\frac{1}{a} \in I$. In I, if you take the inverse of integers other than 1, the result (if it exists at all) is not an integer.**

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Vector

Vector is a quantity which has magnitude and direction.

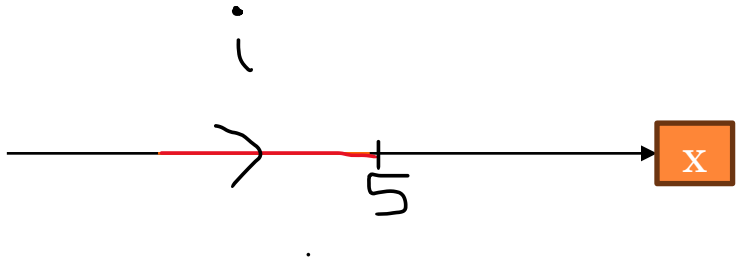
Vector = Magnitude + Direction

If a car is moving 60 Km/hr in the east direction, then 60 is magnitude (scalar), East is the direction.

1-Dimensional Vector:

Let $\vec{v} = 5\,\boldsymbol{i}$, $\boldsymbol{i}$ is unit vector along $x - axis$.

5 represents the length of the vector, $\boldsymbol{i}$ represents the direction.

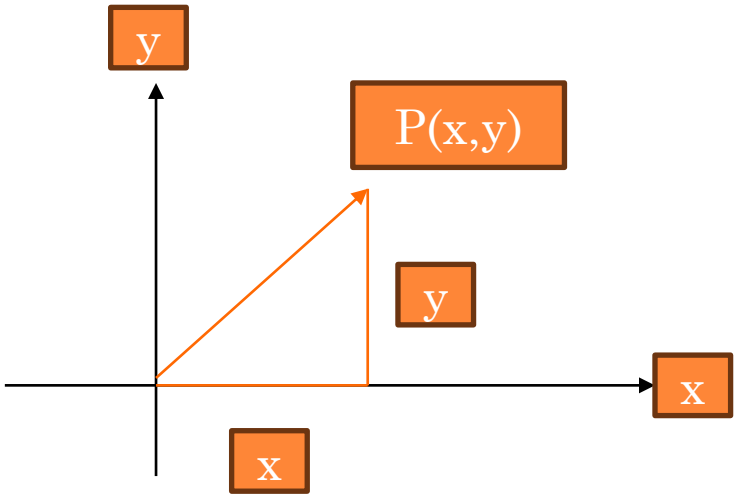


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2-Dimensional Vector:

Let $P(x, y)$ be a point in $\mathbb{R}^2$ −plane. Then the position vector of the point P is represented by $\overrightarrow{OP}$ and is referred as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.



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Example:

Length of $\vec{v} = 4\,\boldsymbol{i} + 3\boldsymbol{j}$

$|\vec{v}| = \sqrt{4^2 + 3^2} = 5 = v(scalar)$

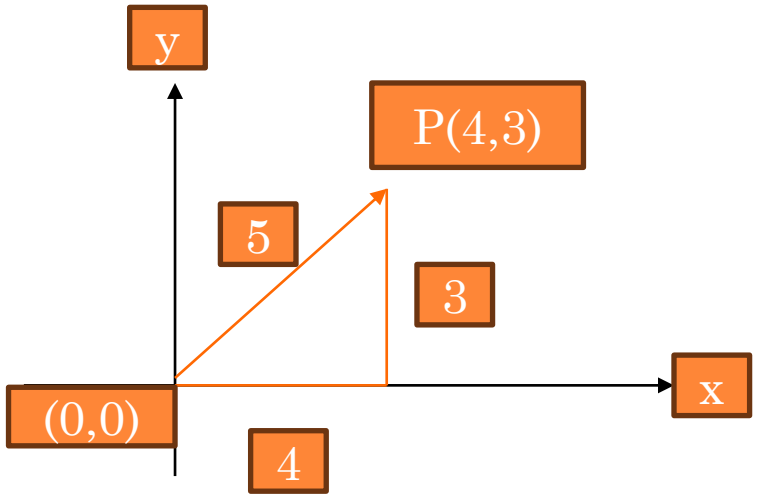
In general:

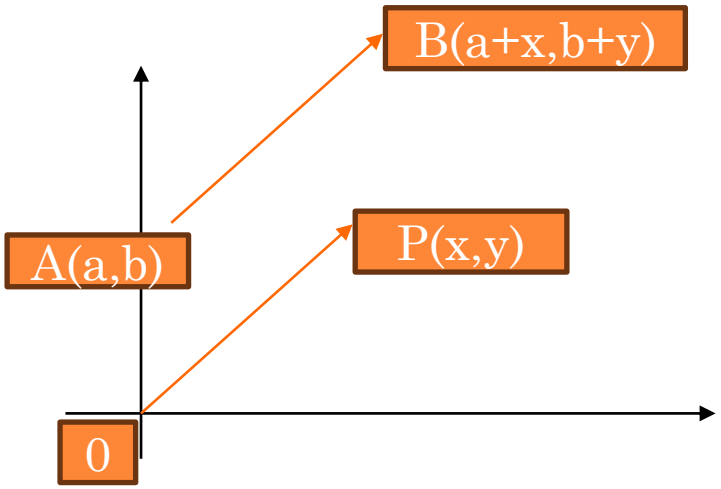
Length of $\vec{v} = x\,\boldsymbol{i} + y\boldsymbol{j}$

$|\vec{v}| = \sqrt{x^2 + y^2} = v$

This is nothing but

Pythagoras Theorem.

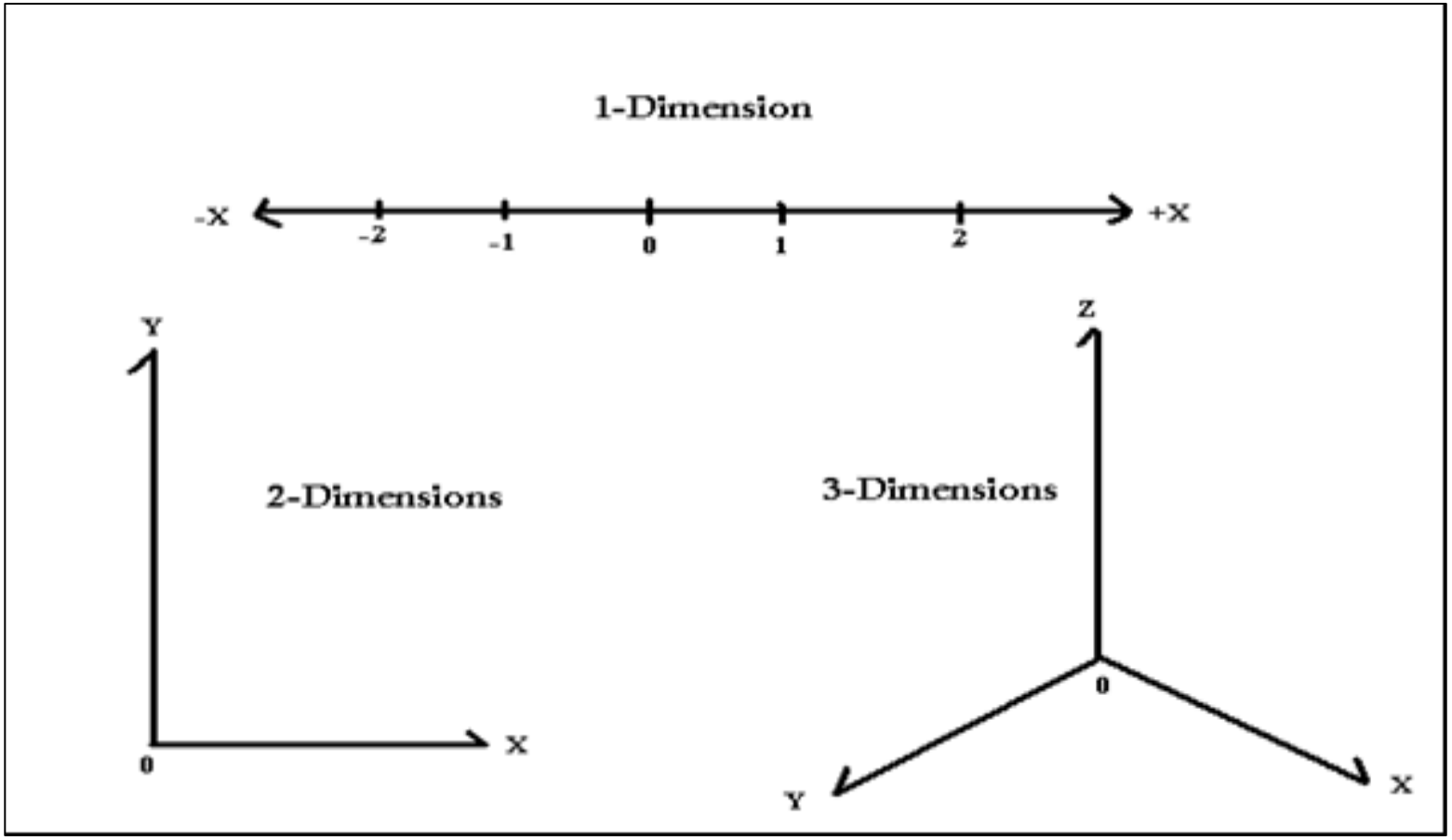




Is $\overrightarrow{AB} = \overrightarrow{OP}$?

YES, $\overrightarrow{AB} = \overrightarrow{OP}$.

If the length (magnitude) of a vector $\vec{u}$ is the same as the length of a vector $\vec{v}$, and they point in the same direction in space, then $\vec{u}$ and $\vec{v}$ are equal vectors.

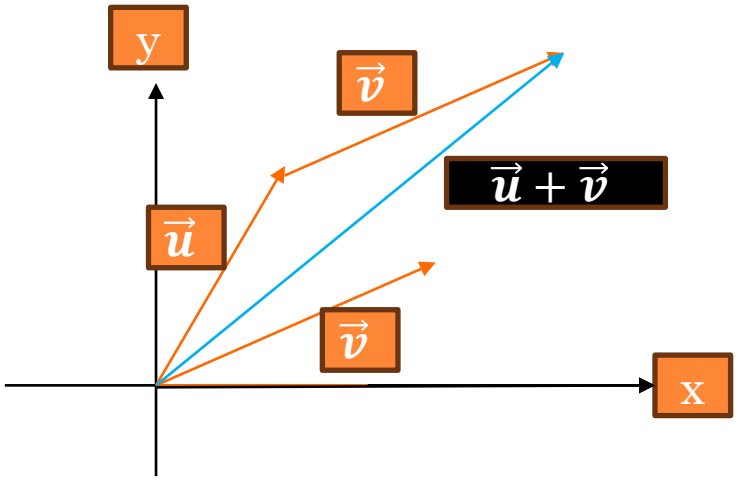

1-Dimension
-X
-2
-1
0
1
2
+X
Y
2-Dimensions
0
X
Z
3-Dimensions
0
Y
X

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ADDITION OF VECTORS:

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.



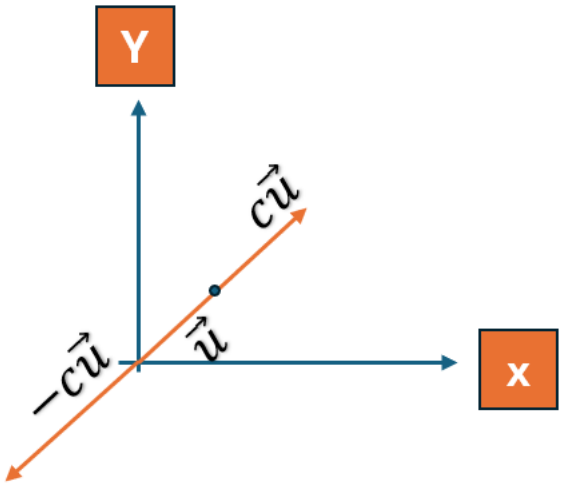
SCALAR MULTIPLICATION OF A VECTOR

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\alpha \in \mathbb{R}$ be any scalar

Then $\alpha\,\vec{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix}$

$\alpha\vec{u}$ magnifies the length in the same direction.

$-\alpha\vec{u}$ magnifies the length in the opposite direction.



VECTOR SPACE

A non empty collection of vectors (in $\mathbb{R}^n$ *where* $n \in \{1,2,3,4 \ldots\}$) along with two operations addition (+) and scalar multiplication ($\cdot$) defined over the field F is called **VECTOR SPACE** $V(+,\cdot)$, if it satisfies the following axioms:

(i) Closure under Addition: For all $\boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} \in V$

Let $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \vdots \\ \boldsymbol{u_n} \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix}$ *where* u_i *and* $v_i \in F$, where F is a field.

(Field may be $R/\ Q/\ C$) *then* $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} \boldsymbol{u_1} + \boldsymbol{v_1} \\ \boldsymbol{u_2} + \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{u_n} + \boldsymbol{v_n} \end{bmatrix}$ → Field addition

↓ Vector addition

(ii) Closure under Scalar Multiplication:

For all u$\in V$ and $\alpha \in F$, $\alpha u \in V$ (Field may be $\mathbb{R}/\ Q/\ C$)

$$\boldsymbol{Let\ u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \vdots \\ \boldsymbol{u_n} \end{bmatrix}, then\ \boldsymbol{\alpha.u} = \begin{bmatrix} \boldsymbol{\alpha.u_1} \\ \boldsymbol{\alpha.u_2} \\ \vdots \\ \boldsymbol{\alpha.u_n} \end{bmatrix} \in V$$

(iii) Additive Identity: There exists a $\overrightarrow{0} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}_n \in \boldsymbol{V}$

Also, Commutativity of Addition, Associativity of Addition, Existence of the Additive Inverse, Existence of the Multiplicative Identity, Associativity of Scalar Multiplication, Distribution of Elements to Scalars, Distribution of Scalars to Elements must hold.

If the set of vectors in $V(+,.)$ satisfy the above axioms, then V is called **vector space over a field F**.

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Examples

1. Write the magnitude of these vectors

a) $\vec{v} = 4\,i + 3j$

$$|v| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

a) $\vec{v} = 2\,i - j$

$$|v| = \sqrt{(2^2 + (-1)^2)} = \sqrt{5}$$

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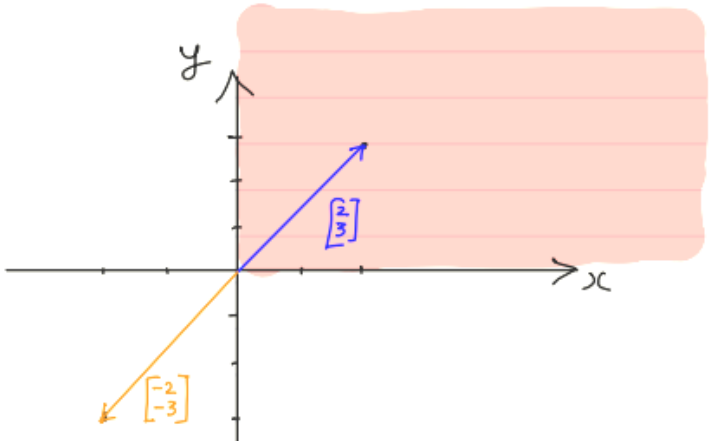
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EXAMPLE PROBLEMS

1. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \right\}$ a vector space over $\mathbb{R}$ with usual addition and scalar multiplication?

Ans: No.

Since, if we consider $\alpha = -1$, then $\alpha . \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin V$.

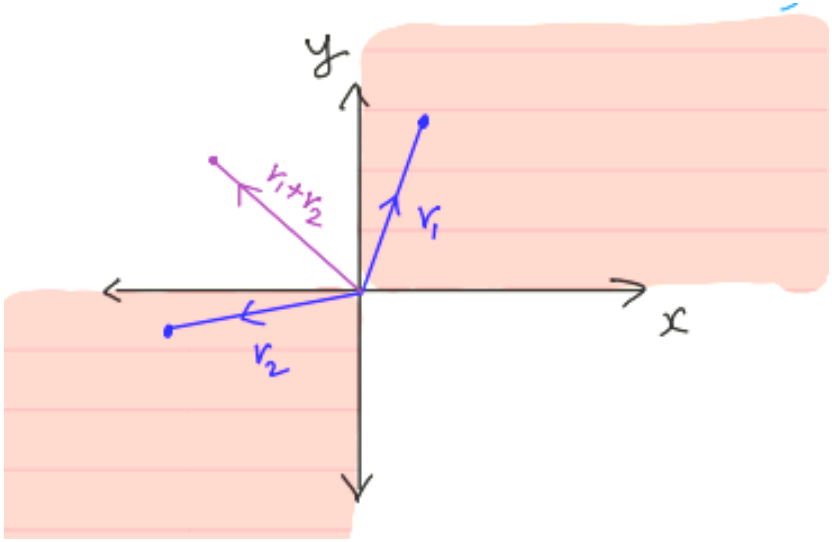


2. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \; or \; x \leq 0, y \leq 0 \right\}$ a vector space over $\mathbb{R}$?

Ans: No.

In the fig., $v_1, v_2 \in V$ but not $v_1 + v_2$

Let $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin V$



3. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle|\ x \geq 0 \text{ and } y \geq 0 \text{ or } x \leq 0, y \leq 0 \text{ or } x \leq 0 \text{ and } y \geq 0 \right\}$ a vector space over $\mathbb{R}$?

Let $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in V$

Now multiply by the scalar $c = -1$

$$cv = -1.\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Here, $x = 1 > 0$ and $y = -1 < 0$. This vector belongs to the fourth quadrant, which is not included in V.

4. $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ with usual matrix addition and scalar multiplication defined by

i) $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

ii) $\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$ for every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$ and $\alpha \in \mathbb{R}$.

In general, $\mathbb{R}^{m \times n}$, set of real $m \times$ n matrices is a vector space over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

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5. Verify whether the Set of all polynomials of degree ≤ 2

$P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$

i) $(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2$

ii) $\alpha(a_1 + b_1x + c_1x^2) = \alpha a_1 + (\alpha b_1)x + (\alpha c_1)x^2$

for every $(a_i + b_ix + c_ix^2) \in P_2$ and $\alpha \in R$ where $i = 1,2$.

Also, zero vector is $0 + 0x + 0x^2$. Hence, it is a vector space over $\mathbb{R}$.

6. Check whether the following sets form a vector space over $\mathbb{R}$:

i) Let $M = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ with usual matrix addition and scalar multiplication.

ii) $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \right\}$ with usual matrix addition and scalar multiplication.

SELF STUDY PROBLEMS

1. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \right\}$ a vector space over $\mathbb{R}$ with usual addition and scalar multiplication?

2. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \; or \; x \leq 0, y \leq 0 \right\}$ a vector space over $\mathbb{R}$?

3. Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \; and \; y \geq 0 \; or \; x \leq 0, y \leq 0 \; or \; x \leq 0 \; and \; y \geq 0 \right\}$ a vector space over $\mathbb{R}$?

4. $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ with usual matrix addition and scalar multiplication defined by

i) $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

ii) $\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$ for every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$ and $\alpha \in \mathbb{R}$.

In general, $\mathbb{R}^{m \times n}$, set of real $m \times$ n matrices is a vector space over $\mathbb{R}$ under usual matrix addition and scalar multiplication.

5. Verify whether the Set of all polynomials of degree ≤ 2
$P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$

6. Check whether the following sets form a vector space over $\mathbb{R}$:

i) Let $M = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ with usual matrix addition and scalar multiplication.

ii) $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \right\}$ with usual matrix addition and scalar multiplication.

THINK!

- Which of the following is an example of a vector space?
  a) The set of all polynomials of degree exactly 2 with real coefficients
  b) The set of all 2×2 matrices with real entries
  c) The set of positive real numbers under addition
  d) The set of natural numbers under addition

b) The set of all 2×2 matrices with real entries

THINK!

- The zero vector in a vector space is important because:

A) It is the only vector that can be part of a basis.
B) It is needed to satisfy the additive identity property of vector spaces.
C) It can be generated by any nonzero scalar multiplication.
D) It is never used in vector space computations.

B) It is needed to satisfy the additive identity property of vector spaces.

The additive identity (zero vector) is required by the definition of a vector space.

THINK!

- If u and v are vectors in a vector space V, then $u - v$ is always in V.

  True or False?

True

Since V is a vector space, it satisfies the following properties:
1.Closure under scalar multiplication: If v∈ V , then $(-1)v \in V$.
2.Closure under addition: If $(u \in V)$ and $(w \in V)$, then $(u + v \in V)$.
Given $(u, v \in V)$,

$$[\, u - v = u + (-v) = u + (-1)v \,]$$

Since $((-1)v \in V)$ and (V) is closed under addition, we get

$$[\, \{ u - v \in V \} \,]$$

THINK!

- A vector space over $\mathbb{R}$ implies:
  a) Scalars are real numbers
  b) Vectors are real numbers
  c) The zero vector is always 1
  d) Vectors can only be integers

a) Scalars are real numbers

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VECTOR SUBSPACE

A subset W of a vector space V over the real numbers $\mathbb{R}$ is called a **subspace** of V if W is a vector space over $\mathbb{R}$ with the operation of addition, scalar multiplication and additive identity defined on V.

Let V be a vector space over a field F and W be a non empty subset of V. Then, W is said to be a subspace of vector space V if :

1. $u, v \in W \; then \; u + v \in W$.

2. $u \in W \;\; and \; \alpha \in F \;\; then \;\; \alpha u \in W$.

3. $0 \in W$ (is a consequence of (2))

Then, W is said to be subspace of V.

VECTOR SUBSPACE

Example:

1. Let us obtain all subspaces of $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x, y \in R \right\}$ under usual binary operations subspace of $\mathbb{R}^2$

i) All of $\mathbb{R}^2$ (Improper Subspace)

ii)S=$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, zero vector only (Trivial subspace)

iii) Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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Example:

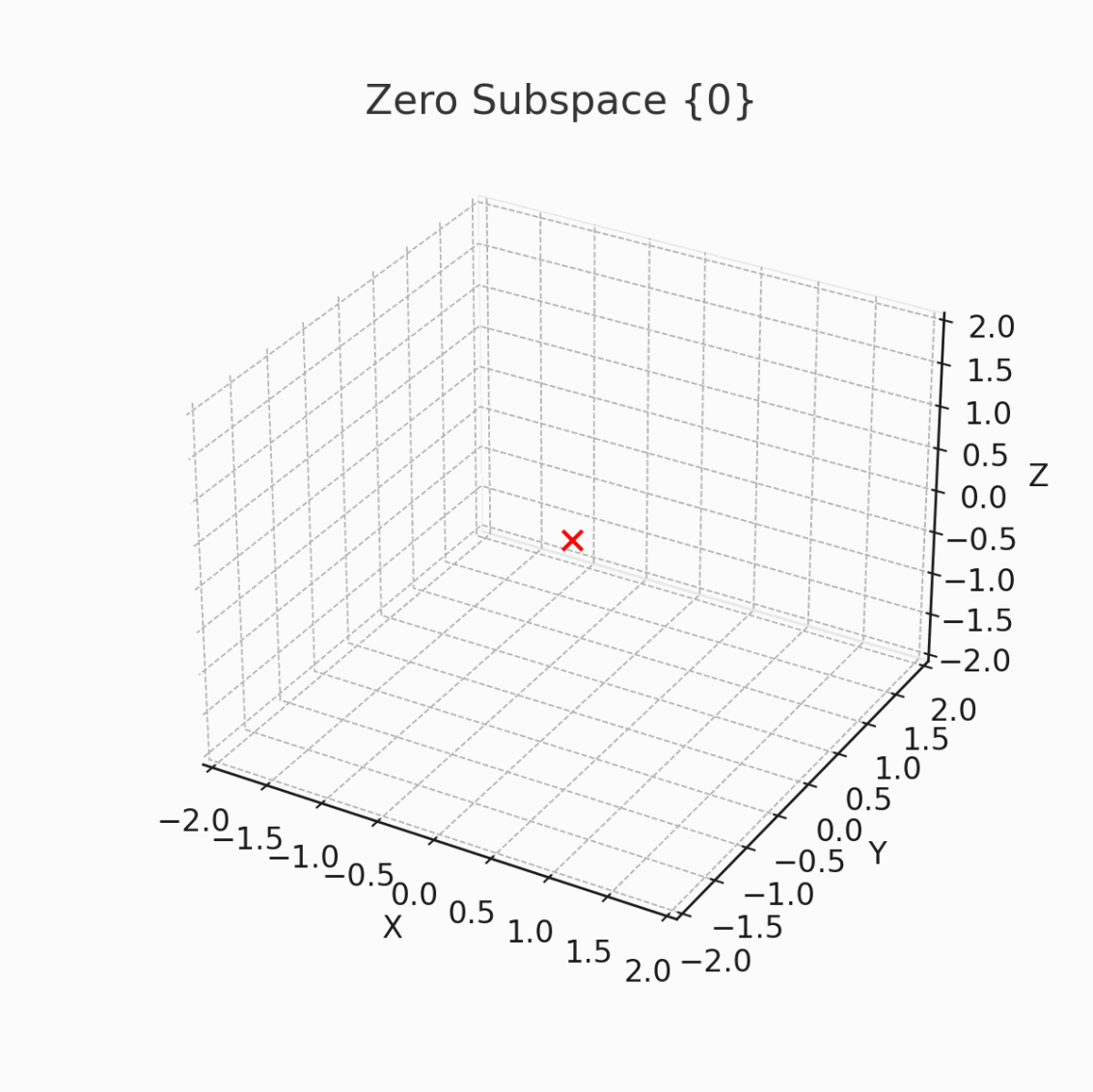
2. Find all subspaces of $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$ under usual binary operations.

i) All of $\mathbb{R}^3$ (Improper Subspace)

ii) Zero vector only that is $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ (Trivial subspace)

iii) All lines through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

iv) All planes through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$


Zero Subspace {0}
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
Z
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
X
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
Y

THINK!

- Let V be a vector space over $\mathbb{R}$, and $W \subseteq V$ such that for all $v \in W$ and scalar $\alpha \in \mathbb{R}, \alpha \cdot v \in W$. However, W is not closed under addition. What can we conclude about W?

A) W is a subspace of V.
B) W is not a subspace of V.
C) W is a vector space by itself.
D) W must be empty.

B) W is not a subspace of V. (Why?)

Both closure under scalar multiplication and addition are required for subspace

THINK!

- The zero vector alone forms a subspace of any vector space.

A) True
B) False

A) True

VECTOR SUBSPACE

Remarks:

i) Every subspace of a vector space is a vector space.

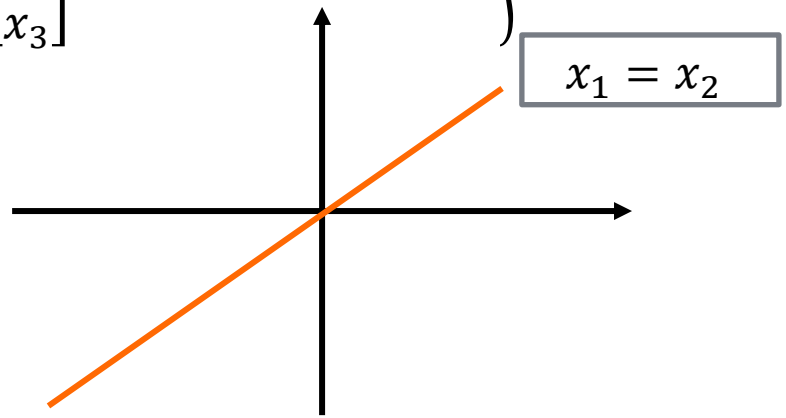
ii) Any vector space V automatically contains two subspaces:

a) The set $\{0\}$, set consisting of only zero vector, is called trivial subspace.

b) V itself is called improper subspace.

EXAMPLES

1. Given $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 = x_2 \; and \; x_i \in \mathbb{R} \right\}$. Verify S is a subspace of $\mathbb{R}^3$.



2. Consider a set of vectors in three-dimensional space, $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \right\}$.

   Determine whether this set forms a vector space. If it does, prove it; if not, provide a counter example for one of the vector space axioms.

3. Consider the set of vectors

$$S = \{[x \ y \ z]^T \mid x, y, z \in \mathbb{R}, 2x + 3y - z = 0\}.$$

Determine whether the set S defined above forms a subspace of $\mathbb{R}^3$. Specifically, check if it satisfies all the conditions & if it is a subspace, prove it. If it is not, provide a counter example for one of the conditions.

4. Verify whether the subset $W = \{[\, x_1 \;\; x_2 \;\; x_3 \,]^T / \;\; 5x_1 + 5x_2 + 5x_3 = 0\}$ of the vector space $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$.

Yes.

Let $u = [\, x_1 \;\; x_2 \;\; x_3 \,]^T \in W$, then $5x_1 + 5x_2 + 5x_3 = 0$. Let $v = [\, y_1 \;\; y_2 \;\; y_3 \,]^T \in W$, then $5y_1 + 5y_2 + 5y_3 = 0$.
Now, $u + v = [\, x_1 + y_1 \quad x_2 + y_2 \quad x_3 + y_3]^T$ with
$5(x_1 + y_1) + 5(x_2 + y_2) + 5(x_3 + y_3) = 5x_1 + 5x_2 + 5x_3 + 5y_1 + 5y_2 + 5y_3 = 0$. Hence, $u + v \in W$.

Let $\alpha \in R$. Then, $\alpha u = \alpha[\, x_1 \;\; x_2 \;\; x_3 \,]^T = [\alpha x_1 \;\; \alpha x_2 \;\; \alpha x_3 \,]^T$ with $5\alpha x_1 + 5\alpha x_2 + 5\alpha x_3 = \alpha(5x_1 + 5x_2 + 5x_3) = 0$. Hence, $\alpha u \in W$.

Also, $[0\ 0\ 0]^T \in W$.

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SELF STUDY PROBLEMS

1. Given $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 = x_2 \; and \; x_i \in \mathbb{R} \right\}$. Verify S is a subspace of $\mathbb{R}^3$.

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4. Verify whether the subset $W = \{[\, x_1 \;\; x_2 \;\; x_3 \,]^T / \;\; 5x_1 + 5x_2 + 5x_3 = 0\}$ of the vector space $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$.

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HOME WORK PROBLEMS

1. Show that the subset $W = \{[x_1\ x_2\ x_3]^T | x_1 + x_2 + x_3 = 0\}$ of the vector space $V_3(R)$ is a subspace of $\mathbb{R}^3$.
2. Determine whether the set $W = \{[a\ b\ c\ d]^T | a + b = 1\}$ is a subspace $\mathbb{R}^4$.
3. Given $S = \left\{\begin{bmatrix} x \\ 1 \end{bmatrix} | \ x \in R\right\}$. Check whether S is a subspace of $\mathbb{R}^2$. Represent the given set geometrically.
4. Determine whether the set of vectors,

$S = \{[x\ y\ z]^T \mid x, y, z \in \mathbb{R},\ \ 10x - 12y + 2z = 0\}$ is a vector space.

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REMARKS

1. If w_1 & w_2 are two subspaces of a vector space V then $w_1 \cap w_2$ is also a subspace of V.
2. If w_1 & w_2 are two subspaces of a vector space V then $w_1 \cup w_2$ is a subspace of V iff $w_1 \subset w_2$ *or* $w_2 \subset w_1$.

LINEAR COMBINATION

Definition: Let V be a vector space over the field F and $v_1, v_2, \ldots, v_n \in V$ then a vector $'u'$ of the form, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = u$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, (some of them may be zero) then $'u'$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$.

Example: Let $[1\ 2\ 1]^T, [-1\ 0\ 2]^T$ *and* $[-1\ 4\ 8]^T$ be three vectors in R^3.

$$2[1\ 2\ 1]^T + 3[-1\ 0\ 2]^T = [-1\ 4\ 8]^T$$

$[-1\ 4\ 8]^T$ is a linear combination of the vectors $[1\ 2\ 1]^T, [-1\ 0\ 2]^T$.

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WORKED EXAMPLES

1. Is $[1\ -3\ 4]^T$ is a linear combination of $[0\ -1\ 5]^T and\ [-2\ 3\ 4]^T$

Solution: $[1\ -3\ 4]^T$ is a linear combination of $[0\ -1\ 5]^T and\ [-2\ 3\ 4]^T$

If $[1\ -3\ 4]^T = \alpha_1[\ 0\ -1\ 5]^T + \alpha_2[-2\ 3\ 4]^T$ holds

To find $\alpha_1 and\ \alpha_2$ such that the above condition holds

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$0\ \alpha_1 - 2\alpha_2 = 1$$

$$-1\ \alpha_1 + 3\alpha_2 = -3$$

$$5\ \alpha_1 + 4\alpha_2 = 4$$

Which represents a system of linear equations of the form $\boldsymbol{A\alpha = B}$

LINEAR COMBINATION

Where $A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 5 & 4 \end{bmatrix}$, $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

On solving the above system of equations by matrix method, we get

$$\begin{bmatrix} -1 & 3: & -3 \\ 0 & -2: & 1 \\ 0 & 0: & -3 \end{bmatrix}$$

$$\rho(A) \neq \rho[A:B)$$

$$=> \textit{there is no such } \alpha_1 \textit{ and } \alpha_2 \textit{ such that}$$

$[1 \ -3 \ 4]^T = \alpha_1[0 \ -1 \ 5]^T + \alpha_2[-2 \ 3 \ 4]^T$ holds

Therefore it is not a linear combination.

2. Is the vector $[3 \ 5 \ 2]^T$ as a linear combination of the vectors $[1 \ 1 \ 0]^T, [2 \ 3 \ 0]^T$, $[0 \ 0 \ 1]^T$ of $V_3(\mathbb{R})$.

Solution: $\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = 2$

$\therefore [3 \ 5 \ 2]^T = -1[1 \ 1 \ 0]^T + 2[2 \ 3 \ 0]^T + 2[0 \ 0 \ 1]^T$. Yes, it is a linear combination.

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SELF STUDY PROBLEMS

1. Verify the vector $v = [4\ \ 2\ \ 1]^T$ is a linear combination of the vectors

   $u_1 = [1\ \ -3\ \ 1]^T, u_2 = [0\ \ 1\ \ 2]^T, u_3 = [5\ \ 1\ \ 57]^T$.

2. Verify the vectors $[2\ \ -1\ \ -8]^T$ as a linear combination of the vectors

   $[1\ 2\ 1]^T, [1\ 1\ \ -1]^T, [4\ 5\ \ -2]^T$ in $\mathbb{R}^3$.

HOMEWORK PROBLEMS

1. Verify the vectors $[1 \ -2 \ -5]^T$ as a linear combination of the vectors $[1 \ 1 \ 1]^T, [1 \ 2 \ 3]^T, [2 \ -1 \ 1]^T$ in $\mathbb{R}^3$.

Solution: $[1 \ -2 \ -5]^T = -6[1 \ 1 \ 1]^T + 3[1 \ 2 \ 3]^T - [2 \ -1 \ 1]^T$

2. For which value of k will the vector $[1 \ -2 \ k]^T$ in $\mathbb{R}^3$ be a linear combination of the vectors $[3 \ 0 \ -2]^T$ and $[2 \ -1 \ -5]^T$.

Solution: $k = -8$

THINK!

- Let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$.
  Which of the following is a valid linear combination of v_1, v_2, v_3?
  a) $2v_1 + 3v_2 - v_3$
  b) $v_1 \cdot v_2$ (component-wise multiplication)
  c) $v_1/2 + v_2/3$
  d) None of the above

a) $2v_1 + 3v_2 - v_3$

THINK!

- What is a linear combination of vectors v_1 and v_2 in $\mathbb{R}^2$?

A) $v_1 + v_2$
B) $c_1 \cdot v_1 + c_2 \cdot v_2$ where c_1 and c_2 are scalars
C) $v_1 \times v_2$ (cross product)
D) The sum of the squares of v_1 and v_2

B) $c_1 \cdot v_1 + c_2 \cdot v_2$ where c_1 and c_2 are scalars

THINK!

- Which of the following is NOT a linear combination of $v_1 = (1, 0)$ and $v_2 = (0, 1)$ in $\mathbb{R}^2$?

A) $(3, 2)$
B) $(-1, 0)$
C) $(0, 0)$
D) $(1, 1, 0)$

D) $(1, 1, 0)$

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1. Express $[3\ 5]^T$ as a linear combination of $[1\ 1]^T, [1\ 2]^T$.

Answer: $Let [3\ 5]^T = a[1\ 1]^T + b[1\ 2]^T$

Then: $a + b = 3 \qquad a + 2b = 5$

On solving we get, $[3\ 5]^T = 1[1\ 1]^T + 2[1\ 2]^T$

2. Express $[2\ 3\ 1]^T$ as a linear combination of $[1\ 1\ 0]^T, [1\ 0\ 1]^T, [0\ 1\ 1]^T$.

Answer: Let $[2\ 3\ 1]^T = a[1\ 1\ 0]^T + b[1\ 0\ 1]^T + c[0\ 1\ 1]^T$

Equating: $a + b = 2,\ a + c = 3, b + c = 1$

Solving gives $a = 2, b = 0, c = 1.$

$(2,3,1) = 2[1\ 1\ 0]^T + 0[1\ 0\ 1]^T + 1[0\ 1\ 1]^T$

3. Can we represent any vector in $\mathbb{R}^3$ as a linear combination of $[1\ 0\ 0]^T, [0\ 1\ 0]^T$ and $[0\ 0\ 1]^T$?

Answer: Yes. Let $u = [1\ 0\ 0]^T, v = [0\ 1\ 0]^T, w = [0\ 0\ 1]^T$.

Any vector $(x, y, z) \in R^3$ can be written as:

$$(x, y, z) = xu + yv + zw.$$

LINEAR INDEPENDENT AND DEPENDENT

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if their linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (only trivial solution).

A set of vectors $\{v_1, v_2 \ldots v_n\}$ is **linearly dependent** if there exist coefficients

α_1 , α_2 ... α_3 , not all zero, such that the equation $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ (non-trivial solution), i.e., at least one coefficient is non-zero.

Thus,

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 = 0 \rightarrow$ Linearly independent

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ not all $\alpha_i = 0 \rightarrow$ Linearly dependent

- $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$, not all $\alpha_i = 0 \rightarrow$ Linearly dependent

Lets consider $\alpha_1 v_1 + \alpha_2 v_2 = 0$, not all $\alpha_i = 0$

$$\Rightarrow \alpha_1 \, v_1 = -\alpha_2 v_2 \;\Rightarrow\; v_1 = \frac{-\alpha_2}{\alpha_1} v_2$$

Thus, v_1 is a scalar multiple of v_2.

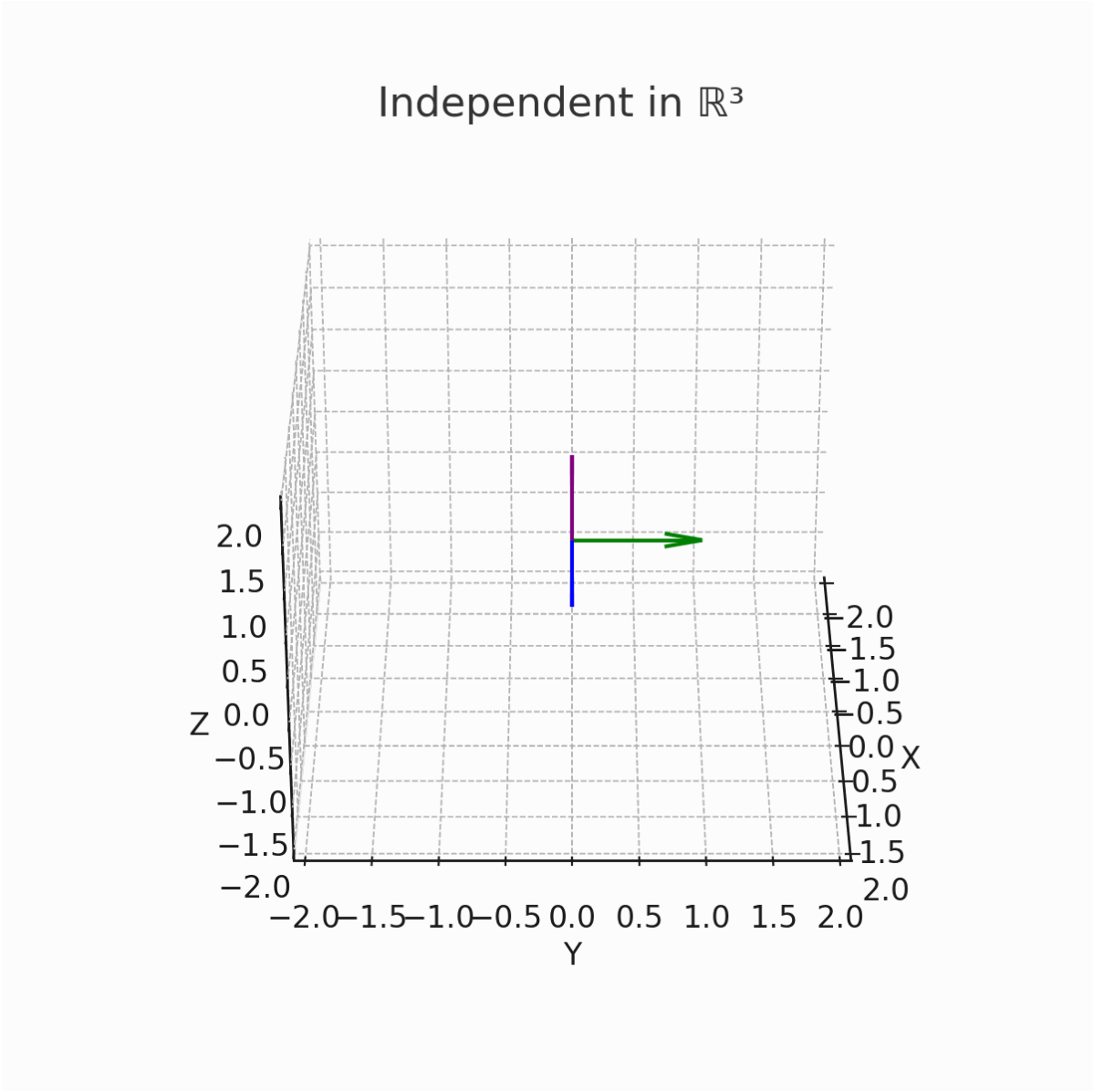
- $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \rightarrow$ Linearly independent

If any one of the $\alpha_n \neq 0$ then $\alpha_1 v_1 + \alpha_2 v_2 \neq 0$

$$\Rightarrow \alpha_1 \, v_1 \neq -\alpha_2 v_2 \;\Rightarrow\; v_1 \neq \frac{-\alpha_2}{\alpha_1} v_2$$

Thus, v_1 is not a scalar multiple of v_2.

Remark: Linearly dependent vectors lie on the same line in $\mathbb{R}^2$ & same plane in $\mathbb{R}^3$.


Independent in $\mathbb{R}^3$
2.0
1.5
1.0
0.5
Z 0.0
−0.5
−1.0
−1.5
−2.0
−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0 1.5 2.0
Y
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
X

WORKED EXAMPLES

1. Examine the vectors $u = [1 \ \ 2 \ \ -1]^T$, $v = [-2 \ \ 0 \ \ 1]^T$, and $w = [3 \ \ -1 \ \ 2]^T$ belonging to the vector space $\mathbb{R}^3$ for linear dependency.

Solution: To find a linear combination of vectors u, v, and w that result in the zero vectors, we need to find coefficients c_1, c_2, and c_3 such that:

$$\alpha_1 u + \alpha_2 v + \alpha_3 w = [0 \ 0 \ 0]^T$$

We have the following vectors:

$$u = [1 \ \ 2 \ \ -1]^T$$

$$v = [-2 \ \ 0 \ \ 1]^T$$

$$w = [3 \ \ -1 \ \ 2]^T$$

Now, let's set up the equations based on the linear combination:

$$\alpha_1 u + \alpha_2 v + \alpha_3 w = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have a system of linear equations:

$$c_1 - 2c_2 + 3c_3 = 0$$

$$2c_1 - c_3 = 0$$

$$-c_1 + c_2 + 2c_3 = 0$$

On solving (use Gauss-elimination),

$$c_1 = 0, c_2 = 0, c_3 = 0$$

In other words, the coefficients are all zero, and the linear combination is:

$$0u + 0v + 0w = [0\ 0\ 0]^T$$

=> Linearly independent.

2. Examine whether the vectors $u = [1\ 2\ 1\ 0]^T, v = [2\ 4\ 2\ 0]^T$ and $w = [3\ 6\ 3\ 0]^T$ are linearly independent in $\mathbb{R}^4$.

Solution: To determine whether the vectors $u = [1\ 2\ 1\ 0]^T, v = [2\ 4\ 2\ 0]^T$, and w $= [3\ 6\ 3\ 0]^T$ are linearly independent in $\mathbb{R}^4$, we need to check if there exist constants a, b, and c, not all equal to zero, such that:

$$au + bv + cw = [0\ 0\ 0\ 0]^T$$

Let's set up and solve a system of equations to find these constants:

$$a[1\ 2\ 1\ 0]^T + b[2\ 4\ 2\ 0]^T + c[3\ 6\ 3\ 0]^T = [0\ 0\ 0\ 0]^T$$

This leads to the following system of equations:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$
$$a + 2b + 3c = 0$$
$$0 = 0$$

Notice that equations 1 and 3 are identical, which means there are only three unique equations in the system. Now, we can simplify the equations:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$
$$0 = 0$$

Let's simplify equations 1 and 2 further:

$$a + 2b + 3c = 0$$
$$2a + 4b + 6c = 0$$

Divide equation 2 by 2:

$$2a/2 + 4b/2 + 6c/2 = 0$$

$$a + 2b + 3c = 0$$

Now, we see that equations 1 and 2 are equivalent, so we have two identical equations:

$$a + 2b + 3c = 0$$

$$a + 2b + 3c = 0$$

Since we have only one equation for three variables (a, b, and c), there are infinitely many solutions that satisfy this system. This indicates that the vectors u, v, and w are linearly dependent in $\mathbb{R}^4$ because we can find non-trivial constants a, b, and c that make the linear combination equal to the zero vector.

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SELF STUDY PROBLEMS

1. Determine whether the vectors $[1\ 2\ 3]^T, [3\ 1\ 2]^T, [2\ 5\ 8]^T$ are linearly dependent

2. Determine whether the vectors $[1\ 4\ 9]^T, [3\ 1\ 4]^T, [9\ 3\ 12]^T$ are linearly dependent

HOMEWORK PROBLEMS

1. Verify that the vectors $v_1 = [1\ 1\ 1]^T, v_2 = [1\ 0\ 1]^T, v_3 = [0\ 1\ 1]^T$ are linearly independent.
2. Verify that the vectors $[1\ 1\ 2\ 4]^{\mathrm{T}}, [2\ -1\ -5\ 2]^T, [1\ -1\ -4\ 0]^T$ and $[2\ 1\ 1\ 6]^{\mathrm{T}}$ are linearly dependent in $\mathbb{R}^4$.

THINK!

- The set $\{(1,0),(2,0)\}$ is linearly independent in $\mathbb{R}^2$. True or False?

False (Why?)

THINK!

- In $\mathbb{R}^3$, any set of four vectors is always:
  a) Linearly independent
  b) Linearly dependent

b) Linearly dependent

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1. Given two non-zero vectors in $\mathbb{R}^3$ that are orthogonal, what can we conclude about their linear independence or dependence? Give reason

Answer: If $u \cdot v = 0$, the vectors are orthogonal. Two non-zero orthogonal vectors cannot be scalar multiples of each other.

They are linearly independent.

2. Determine whether the vectors $u = [1,2]^T, v = [2,4]^T$ are linearly dependent or independent.

Answer: Since $v = 2u$, one vector is a scalar multiple of the other.

The vectors are linearly dependent.

3. Specify the condition for Linear dependency and Linear independency for the linear combination of vectors

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$

Answer:

Linearly independent: Only solution is $c_1 = c_2 = \cdots = c_n = 0.$

Linearly dependent: There exists at least one non-zero coefficient such that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$

LINEAR SPAN OF A SET

Let V be a vector space and $v_1, v_2, v_3, \dots, v_n$ be vectors in V. The set of all linear combinations of $v_1, v_2, v_3, \dots, v_n$ is called the **span** of $v_1, v_2, v_3, \dots, v_n$.

It is denoted by $Span(v_1, v_2, v_3, \dots, v_n)$ or $L(v_1, v_2, v_3, \dots, v_n)$.

In other words, $\boldsymbol{Span(V) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \,|\alpha_1, \alpha_2, \dots, \alpha_n \in R\}}$.

For example, $Span\{\begin{bmatrix}1\\-1\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}\} = \{\alpha_1 \begin{bmatrix}1\\-1\\4\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-1\\0\end{bmatrix} | \alpha_1, \alpha_2 \in \mathbb{R}\}$

and a linear combination,

$$3\begin{bmatrix}1\\-1\\4\end{bmatrix} + 2\begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}7\\-5\\12\end{bmatrix} \in Span\{\begin{bmatrix}1\\-1\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}\}.$$

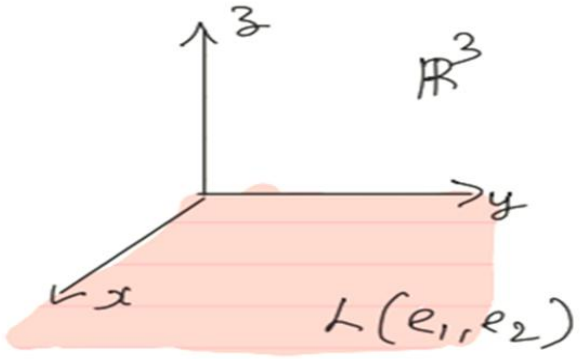
Example: Find the linear span of e_1 *and* e_2 in R^3.

Solution: $L(e_1, e_2) = \{ \alpha_1 e_1 + \alpha_2 e_2 \mid \alpha_1, \alpha_2 \in R \}$

$$= \{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \}$$

$$= \{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \mid \alpha_1, \alpha_2 \in \mathbb{R} \}$$

Thus, the span is the set of all points in the xy-plane.
It is a **subspace of R^3**.



LINEAR SPANNING SET FOR VECTOR SPACE

Definition: The set S = $\{v_1, v_2, v_3, \ldots, v_n\}$ is called **spanning set** for vector space V if and only if span (S) = V.

In other words, if every vector in V can be written as linear combination of S = $\{v_1, v_2, v_3, \ldots, v_n\}$, then S is called a spanning set of V.

Example: The set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ spans $\mathbb{R}^3$.

Solution: $\alpha_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}$

Then S is spanning set for $\mathbb{R}^3$.

THINK!

- What does it mean for a set of vectors to span a vector space V?
  A) Their linear combinations fill a subspace of V
  B) Their linear combinations fill all of V
  C) They are linearly independent
  D) The set is finite

B) Their linear combinations fill all of V

THINK!

- Which of the following sets forms a span in a vector space V?

A) All linear combinations of the set $\{v_1, v_2, \dots, v_n\}$.
B) All orthogonal vectors to v_1.
C) All vectors whose components are multiples of 1.
D) All matrices whose determinant is 1.

A) All linear combinations of the set $\{v_1, v_2, \dots, v_n\}$.

BASIS AND DIMENSION

Definition: The vectors $v_1, v_2, v_3, \ldots, v_n$ form a basis for a vector space V iff

i) $v_1, v_2, v_3, \ldots, v_n$ are linearly independent

ii) Span $(v_1, v_2, v_3, \ldots, v_n) = V$

Definition: The dimension of V is the number of vectors in the basis of V. Denoted as 'dim V'

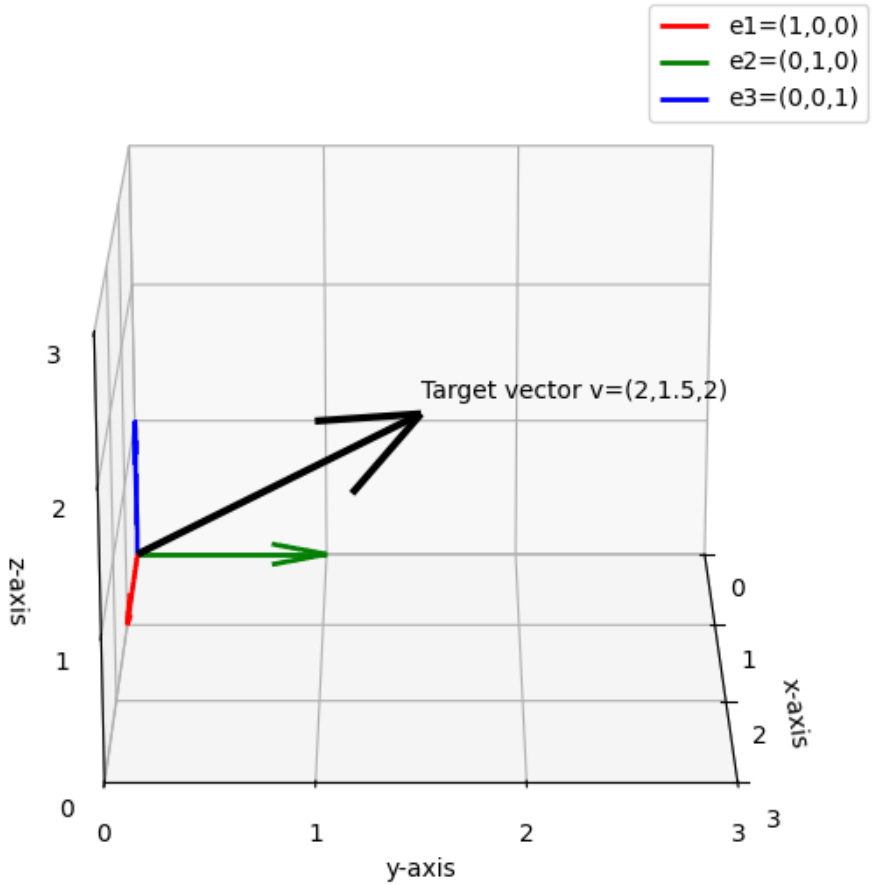
Example:

1. Let $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ then $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$.

Whether $\{e_1, e_2, e_3\}$ forms a basis for $\mathbb{R}^3$?

2. Also, $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$ is basis for $\mathbb{R}^3$.

In fact , there are infinitely many bases for a vector space.


e1=(1,0,0)
e2=(0,1,0)
e3=(0,0,1)
Target vector v=(2,1.5,2)
z-axis
y-axis
x-axis
0
1
2
3

WORKED EXAMPLES

1. Show that the vectors $[1\ 1\ 2\ 4]^T$, $[2\ -1\ -5\ 2]^T$, $[1\ -1\ -4\ 0]^T$, and $[2\ 1\ 1\ 6]^T$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.

Solution:

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 4R_1$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} R_2 \rightarrow -R_2,\ R_2 \rightarrow -\left(\frac{1}{3}\right) R_2,\ R_2 \rightarrow -\left(\frac{1}{2}\right) R_2$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 - R_2,\ R_4 \rightarrow R_4 - R_2$$

The final matrix is the required echelon form. The rank of $A = 2$.
Thus, the given vectors are linearly dependent.

Corresponding to non-zero rows in the final matrix, the rows of A are:

$$(1, 1, 2, 4), (2, -1, -5, 2)$$

These two form a set of linearly independent subset.

The dimension of the subspace spanned by these vectors is 2.
Further, these two vectors form a basis of the subspace.

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2. Find dimension and basis for the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Solution: W.K.T., Dimension = Rank (A)
Therefore, reduce A to echelon form:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(Row operation: } R_3 \rightarrow R_3 - R_1\text{)}$$

Thus, we can write: Dimension (A) = Rank (A) = 2
Basis of Matrix A = number of columns with pivot elements

Hence, the is given by the corresponding original columns:basis for the column space of A
Basis $(A) = \{[1 \;\; 0 \;\; 1]^T, [\,2, 1, 2\,]^T\}$

SELF STUDY PROBLEMS

1. Show that $\{ [1\ 2\ 3]^T, [-2\ 1\ 0]^T, [1\ 0\ 1]^T \}$ is a basis for $\mathbb{R}^3$.

2. Consider the vectors $[1\ 2\ 3]^T, [2\ 4\ 6]^T, [1\ 1\ 1]^T$. Determine if these vectors form a basis for $\mathbb{R}^3$.

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HOMEWORK PROBLEMS

1. Find the dimension and basis of the subspace spanned by the vectors $(2, 4, 2), (1, -1, 0), (1, 2, 1)$, and $(0, 3, 1)$ in $\mathbb{R}^3$.
Answer: Basis = {(1, 2, 1), (0, 3, 1)}; Dimension = 2

2. Find the basis and dimension of the subspace of V spanned by the subset S given by:
i) $S = \{(2,1), (1,-1), (0,2)\}$, $V = \mathbb{R}^2$
Answer: Basis = {(2, 1), (1, −1)}; Dimension = 2

ii) $S = \{(1,1,1), (1,2,3), (-1,0,1)\}$, $V = \mathbb{R}^3$
Answer: Basis = {(1, 1, 1), (1, 2, 3)}; Dimension = 2

iii) $S = \{(1,-2,3,-1), (1,1,-2,3)\}$, $V = \mathbb{R}^4$
Answer: Basis = S; Dimension = 2

1. Specify dimension of the basis $B = \{[1\ -1\ 3]^T, [2\ 6\ 3]^T\}$ of a vector space V.

Answer: $B = \{[1, -1, 3]^{\mathrm{T}}, [2, 6, 3]^{\mathrm{T}}\}$

There are two linearly independent vectors. $\dim(\mathrm{V}) = 2$

2. Find the dimension of the vector space V whose basis is

$$B = \{(1,2,3), (2,1,4)\}$$

Answer: There are 2 linearly independent vectors in the basis.

$\dim(\mathrm{V}) = 2.$

1. Can we represent any vector in $\mathbb{R}^3$ as a linear combination of e_1, e_2 and e_3?

   Yes. (Why?)

Since, e_1, e_2 and e_3 form the standard basis for $\mathbb{R}^3$.

2. Specify dimension of the basis $B = \{[1\ -1\ 3]^T, [2\ 6\ 3]^T\}$ of a vector space V.

$\boldsymbol{dim\ V = 2.}$

3. Let V be a plane passing through the origin in $\mathbb{R}^3$. What is the dimension of V? Justify.

The dimension of V is 2.
This is because a plane in $\mathbb{R}^3$ passing through the origin is a 2-dimensional subspace, which can be spanned by two linearly independent vectors.

4. Consider the standard basis $B_1 = \{[1\ 0\ 0]^T, [0\ 1\ 0]^T, [0\ 0\ 1]^T\}$ of $\mathbb{R}^3$. If we choose another basis B_2 of $\mathbb{R}^3$, is it possible for B_2 to contain 4 vectors? Justify your answer.

No, it is not possible for B_2 to contain 4 vectors.

Observe, $\mathbb{R}^3$ is a 3-dimensional vector space, so the dimension is 3. By definition, every basis of a finite-dimensional vector space must have exactly the same number of vectors as the dimension. Therefore, any basis of $\mathbb{R}^3$ (here, in particular, B_2) can have exactly 3 vectors.

THINK!

- If a set of vectors in $\mathbb{R}^3$ spans the whole space, what is the minimum number of linearly independent vectors required?

A) 1
B) 2
C) 3
D) 4

C) 3

The dimension of $\mathbb{R}^3$ is 3, so at least 3 linearly independent vectors are needed to span the entire space.

MITE
Invent Solutions

Application Problems:

1. Given two emails with feature vectors (6 words, 1 image, 4 links) and (3 words, 2 images, 1 link), weights assigned to these features is given by (0.5, 0.25, 0.25). Assume the emails are labeled as NOT SPAM (0) and SPAM (1).

a) Name the mathematical concept that is involved in weighted sum

b) Evaluate the weighted sum and classify the email as SPAM or NOT SPAM

Solution :

a) The mathematical concept involved in the weighted sum is a linear combination. In this context, the weighted sum can be expressed as a linear combination of the feature vectors, where each feature is multiplied by its corresponding weight.

b) To evaluate the weighted sum for classifying the email as SPAM or NOT SPAM:

Let's denote the feature vector for an email as $X = (x_1, x_2, x_3)$ where x_1 is the number of words, x_2 is the number of images, and x_3 is the number of links.
Given the weights (w_1, w_2, w_3) for word count, image count, and link count, respectively,

the vector representation of the weighted sum for an email can be calculated as follows:
Weighted Sum $= x_1 . w_1 + x_2 . w_2 + x_3 . w_3$

The given weights are (0.5, 0.25, 0.25)

Email 1: (6 words, 1 image, 4 links)
Email 2: (3 words, 2 images, 1 link)

The weighted sum can be calculated as follows:

Email1:

Weighted sum = (6 words * 0.5) + (1 image * 0.25) + (4 links * 0.25)
Weighted Sum=3+0.25+1
Weighted Sum=4.25

Email 2:

Weighted sum = (3 words * 0.5) + (2 image * 0.25) + (1 links * 0.25)
Weighted Sum=1.5+0.5+0.25
Weighted Sum=2.25

MITE

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If $\text{probability}(\text{algorithm}(\text{weighted sum})) \geq 0.5$ SPAM

[because we have labelled SPAM(1)]

If $\text{probability}(\text{algorithm}(\text{weighted sum})) < 0.5$ NOT SPAM

[because we have labelled NOTSPAM(0)]

NOTE: Algorithms will be taught in ADVANCED LINEAR ALGEBRA

MITE
Invent Solutions

Self study Problems:

1. Sentiment Analysis of Movie Reviews:

A dataset containing movie reviews (50 positive, 30 negative, 10 neutral sentiment) along with their corresponding sentiments (positive/ negative/ neutral sentiment) and weights assigned to these sentiments is given by (0.5, 0.5, 0). Assume the reviews are labeled as POSITIVE (1) and NEGATIVE (0).

a. Name the mathematical concept that is involved in weighted sum

b. Evaluate the weighted sum and classify the review as POSITIVE and NEGATIVE

MITE
Invent Solutions

Solution:

a) The mathematical concept involved in weighted sum is linear combination. In linear algebra, a linear combination of variables is the sum of these variables each multiplied by a weight or coefficient.

b) To evaluate the weighted sum and classify the review as POSITIVE or NEGATIVE

Let's denote the feature vector for reviews as $X = (x_1, x_2, x_3)$ where x_1 is the positive reviews , x_2 is the negative reviews , and x_3 is the neutral sentiments .

Given the weights (w_1, w_2, w_3) for positive, negative and neutral sentiments respectively

the vector representation of the weighted sum for reviews can be calculated as follows:

Weighted Sum $= x_1 . w_1 + x_2 . w_2 + x_3 . w_3$

The given weights are (0.5, 0.5, 0)

Weighted sum = (positive * 0.5) + (negative * 0.25) + (neutral * 0)
Weighted Sum=50*0.5+30*0.5+0
Weighted Sum=40

If $\text{probability}(\text{algorithm}(\text{weighted sum})) \geq 0.5$ POSITIVE

[because we have labelled POSITIVE(1)]

If $\text{probability}(\text{algorithm}(\text{weighted sum})) < 0.5$ NEGATIVE

[because we have labelled NEGATIVE(0)]

NOTE: Algorithms will be taught in ADVANCED LINEAR ALGEBRA

INNER PRODUCT SPACE

An inner product space is a vector space where each pair of vectors has a defined "inner product," often denoted as $\langle u, v \rangle$.

Given $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, we define the inner product (dot product) of u and v as:

$$u \cdot v = < u, v > = u^T v = v^T u = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = < v, u >$$

Note: $< u, u > = ||u||^2$ represents the norm or length of the vector.

If v is a nonzero vector in R^n, then the vector,

$$\hat{v} = \frac{v}{||v||}$$

is called the **unit vector** in the direction of v.

The process of finding the unit vector in the direction of v is called **normalizing the vector v**.

GEOMETRIC INTERPRETATION:

1. Angle and length Preservation:

- Inner product operation provides a way to measure angles between vectors and defines the notion of length or magnitude of vectors.
- Inner product allows you to determine the angle between vectors and ascertain whether they are orthogonal (at right angles) or aligned.

2. Orthogonality:

- Orthogonal vectors have an inner product of zero. Geometrically, this means they are perpendicular in space.

NOTE: This inner product function takes in two vectors from the space and produces a scalar value.

WORKED EXAMPLES

Example 1: Consider the vectors $u = [1\ 2\ 4]^T, v = [2\ -3\ 5]^T, w = [4\ 2\ -3\ ]^T$ in $\mathbb{R}^3$. Find: $\langle u, v\rangle, \langle v, w\rangle$

Solution: $< u, v >= (1.2) + (2.-3) + (4.5) = 16$ and

$< v, w >= (2.4) + (-3.2) + (5.-3) = -13$

Example 2: Verify the orthogonality of the vectors: $v = [3 - 1\ 2]^T and\ w = [1\ 3\ 0]^T$.

Solution: $\langle v, w\rangle = (3 \cdot 1) + (-1 \cdot 3) + (2 \cdot 0) = 3 - 3 + 0 = 0.$
The dot product is 0, so v and w are orthogonal.

3. Consider the vectors: $u = [1\ 2\ 4]^T, v = [2\ -3\ 5]^T, w = [1\ 1\ -1]^T$ in $\mathbb{R}^3$.
Find:

1. $\langle u, v \rangle$
2. $\langle v, w \rangle$
3. Check whether u and v, v and w, w and u are orthogonal.

4. Consider the vectors: $u = [2\ 2\ 1]^T, v = [-1\ 4\ 2]^T, w = [3\ 2\ -1]^T$ in $\mathbb{R}^3$.
Find:

1. $\langle u, v \rangle$
2. $\langle v, w \rangle$
3. Check whether u and v, v and w, w and u are orthogonal.

THINK!

- In an inner product space, the inner product of a vector with itself is always positive, unless the vector is zero.

A) True
B) False

A) True

THINK!

- Which of the following is a correct statement about orthogonality in an inner product space?

A) Two vectors u and v are perpendicular if $\langle u, v \rangle = 1$.
B) Two vectors u and v are perpendicular if $\langle u, v \rangle = 0$.
C) Two vectors u and v are orthogonal only if they are linearly dependent.
D) Orthogonality is only defined in $\mathbb{R}^2$.

B) Two vectors u and v are perpendicular if $\langle u, v \rangle = 0$.

THINK!

- In an inner product space, the inner product of any vector v with the zero vector 0 is:

A) Equal to v itself
B) Always 1
C) Always 0
D) Undefined

C) Always 0 (why?)

THINK!

- True or False: $\langle v, v \rangle = 0$ if and only if $v = 0$ (the zero vector).

A) True

1. Check whether the dot product of $u = [1,0]^T$ and $v = [0,1]^T$ is zero.

Answer: $u \cdot v = (1)(0) + (0)(1) = 0$

$u \cdot v = 0$. Hence, the vectors are orthogonal.

2. Check orthogonality of vectors $[1\ 0\ 0]^T$, $[0,1,0]^T$

Answer: $(1)(0) + (0)(1) + (0)(0) = 0$

Yes. The vectors are orthogonal.

3. Check whether the vectors $u = [1\ 2\ 0]^T, v = [2\ -1\ 0]^T$ are orthogonal. Answer: $u \cdot v = (1)(2) + (2)(-1) + (0)(0) = 2 - 2 = 0$ The vectors are orthogonal

4. Given a non-zero vector v that is not of unit length. Is it possible to construct a unit vector in the same direction as v?

Answer: Yes. For any non-zero vector v, the unit vector in its direction is $\hat{v} = \frac{v}{||v||}$

where $||v|| = \sqrt{(v \cdot v)}$.

5. Find the inner product of $u = [2\ \ 1\ \ 3]^T$, $v = [1\ 2\ 1]^T$

Answer: $\langle u, v \rangle = 2(1) + 1(2) + 3(1)$

$= 2 + 2 + 3$

$= 7$

- Given two non-zero vectors in $\mathbb{R}^3$ that are orthogonal, what can we conclude about their linear independence or dependence? Justify.

Two non-zero vectors in $\mathbb{R}^3$ that are orthogonal are always linearly independent.

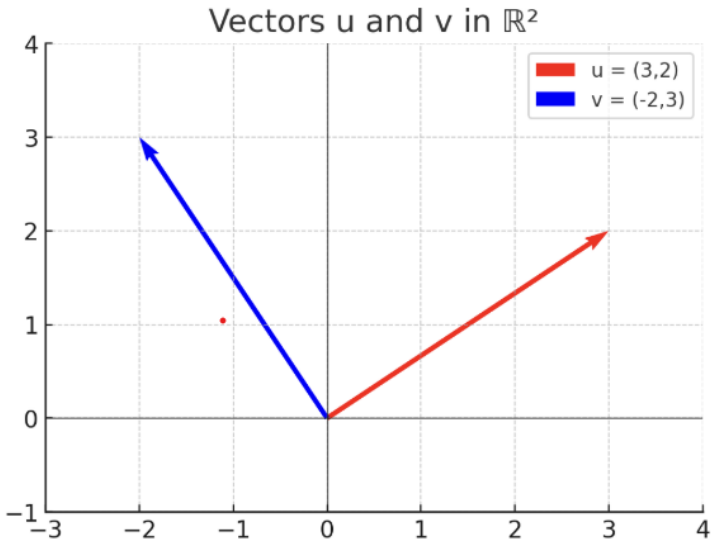
If they were linearly dependent, one vector would be a scalar multiple of the other, leading to non-zero dot product (i.e., not orthogonal).

- Given a non-zero vector v that is not of unit length. Is it possible to construct a unit vector in the same direction as v?

Yes, it is possible.
To construct a unit vector (say u) in the direction of v, divide v by its magnitude, i.e., $u = \frac{v}{||v||}$.

- Consider two non-zero vectors $u = [3\ 2]^T$ *and* $v = [-2\ 3]^T$ whose dot product is zero. Represent this idea geometrically.

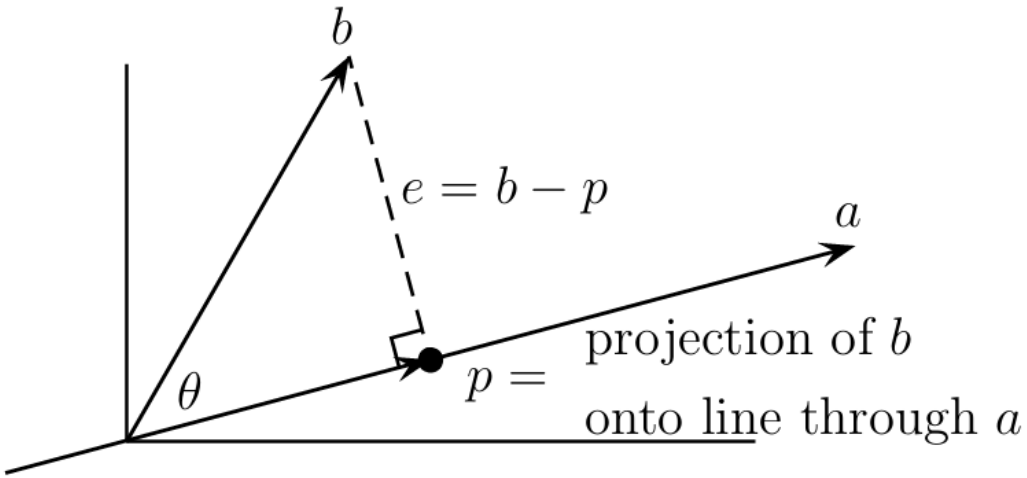
Two non-zero vectors with a dot product of zero are perpendicular to each other. Geometrically, one can observe:



PROJECTIONS

Suppose we want to find the distance from a point 'b' to the line in the direction of the vector a. We are looking along that line for the point p closest to b. Observe, the line connecting b to p (the dotted line in Figure below) is perpendicular to a.

Even though a and b are not orthogonal, the distance problem automatically brings in orthogonality!



The situation is the same when we are given a plane (or any subspace S) instead of a line. Again the problem is to find the point p on that subspace that is closest to b.

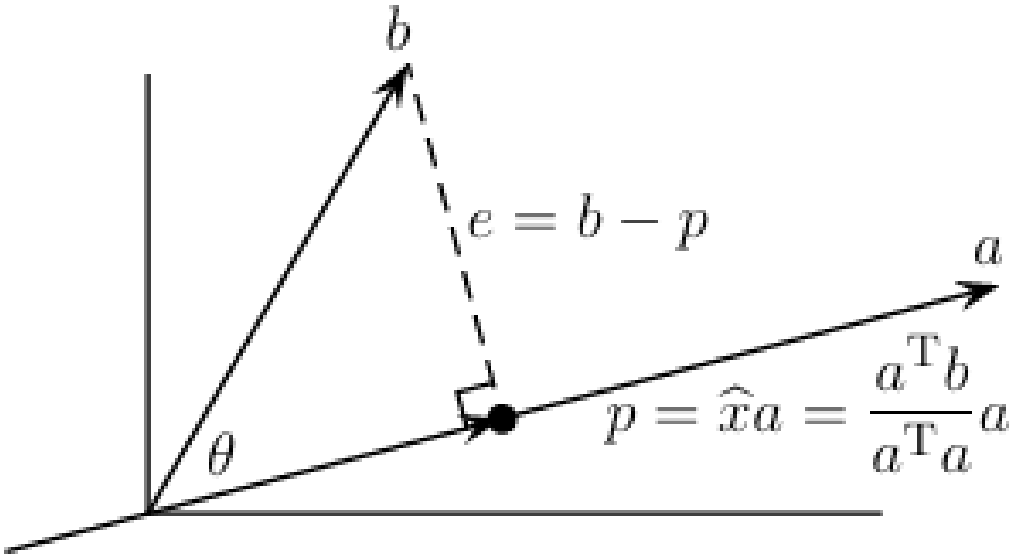
This point p is the projection of b onto the subspace.

A perpendicular line from b to S meets the subspace at p.

Geometrically, that gives the distance between points b and subspaces S.

The projection of the vector ‘b’ onto the line in the direction of ‘a’ is:

$$p = \frac{a^T b}{a^T a} a$$



One can rewrite p as: $p = a \frac{a^T b}{a^T a}$.

i.e., $p = Pb$, where P is the Projection Matrix, given by $P = \frac{aa^T}{a^T a}$.

Thus, the projection of the vector 'b' onto the line in the direction of 'a' is:

$$\mathbf{p = Pb,\ where\ } P = \frac{aa^T}{a^T a}.$$

Note:

The Projection Matrix has the following properties:

1. P is a symmetric matrix.
2. Its square is itself: $P^2 = P$ (Idempotent matrix)

WORKED EXAMPLES

1. Project $b = \begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the line through $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$ to get p. Show that the obtained projection matrix P is symmetric and idempotent.

Solution: The projection of the vector 'b' onto the line through 'a' is:

$$\text{p} = \text{Pb, where } P = \frac{aa^T}{a^T a}.$$

Observe, $a^T a$=3 and $aa^T = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$.

Thus, $P = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$.

Therefore, $p = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$.

Clearly, P is symmetric (since $P = P^T$) and idempotent (since $P^2 = P$).

SELF STUDY PROBLEMS

2. Use the Euclidean inner product in $\mathbb{R}^3$ to find the orthogonal projection of $u = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$ onto $v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

3. Find the matrix P that projects onto the line through $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Show that the obtained matrix P is symmetric and idempotent.

4. A drone is flying in a 3D space, and its current direction is given by the vector $u = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$. The drone needs to adjust its movement to be as close as possible to a specific direction represented by the vector $v = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

Find the projection of u onto v, to determine the closest alignment of the drone's direction with the target direction.

HOMEWORK PROBLEMS

1. Project the vector b onto the line through a, to find p and P. Check whether $e = b - p$ is perpendicular to a.

i. $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \; a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

ii. $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \; a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}.$

Show that the above obtained P is symmetric and idempotent.

THE GRAM-SCHMIDT PROCESS

Suppose we are given three independent vectors a, b, c. Now we propose to find a way to make them orthonormal (orthogonal unit vectors), say q_1, q_2, q_3.

Step 1: Take $q_1 = \frac{a}{||a||}$.

Step 2: Let B = $b - (q_1{}^T b)q_1$ and then, $q_2 = \frac{B}{||B||}$.

Step 3: Let C = $c - (q_1{}^T c)q_1 - (q_2{}^T c)q_2$ and then $q_3 = \frac{C}{||C||}$.

In case there is a fourth vector, the procedure is continued similarly, by subtracting away its components in the direction of q_1, q_2, q_3.

WORKED EXAMPLES

1. Suppose the independent vectors are a, b, c given by: $a = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $b = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $c = \begin{bmatrix}2\\1\\0\end{bmatrix}$. Find the corresponding orthonormal vectors by Gram- Schmidt process.

Solution: Let $q_1 = \frac{a}{||a||} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$.

Let B $= b - (q_1{}^T b)q_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}$ and $q_2 = \frac{B}{||B||} = \sqrt{2}\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}$

Further, C $= c - (q_1{}^T c)q_1 - (q_2{}^T c)q_2 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} - \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and

$q_3 = \frac{c}{||C||} = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Thus, q_1, q_2, q_3 are the corresponding orthonormal vectors by Gram- Schmidt process.

SELF STUDY PROBLEMS

2. In a wireless communication system, three independent signal waveforms (vectors in a signal space) are to be transmitted simultaneously over the same frequency band. The signals are represented by the following vectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Convert these signals into an orthonormal set, which will help minimize interference of signals and ensure that each signal can be accurately detected and demodulated at the receiver.

APPLICATIONS

Vector spaces have wide applications across various fields due to their flexibility in modeling and analyzing different phenomena. Here are a few important applications:

1.Physics and Engineering: In physics for describing quantities like forces, velocities, and electric/magnetic fields. In engineering, for modeling physical systems, such as the analysis of electrical circuits or mechanical structures.

2.Computer Graphics and Computer Vision: In computer graphics, vectors are used to represent positions, directions, and transformations in 2D and 3D spaces. In computer vision, vectors represent features extracted from images, aiding in tasks like object recognition and tracking.

APPLICATIONS

3. Quantum Mechanics: Quantum mechanics extensively uses vector spaces to model states of particles and systems. The wave functions that describe these states often exist in complex vector spaces, allowing for calculations of probabilities and behavior of particles.

4. Economics and Finance: Vector spaces are employed in economics to model preferences, resources, and allocations. In finance, they're used to represent portfolios, asset returns, and risk analysis, aiding in portfolio optimization and risk management.

APPLICATIONS

5. Machine Learning and Data Analysis: Vector spaces play a crucial role in machine learning and data analysis. Vectors represent features in datasets, enabling algorithms to learn patterns, classify data, and make predictions. Techniques like support vector machines and neural networks operate in vector spaces.

6. Control Systems and Robotics: In control systems, vectors help describe states, inputs, and outputs of dynamic systems. In robotics, they're used to represent positions, orientations, and movements of robots, aiding in path planning and control algorithms.

7. Signal Processing and Communication: Vectors are used to represent signals in various domains (time, frequency), facilitating analysis and processing of audio, images, and other types of signals. In communication systems, vector spaces model signal spaces, encoding, and decoding schemes.

These applications demonstrate the versatility and significance of vector spaces across diverse disciplines, providing a powerful framework for understanding and solving complex problems.

THANK YOU